Improved approximation bounds for the dominating set and the vertex cover in power-law graphs

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ABSTRACT

In this work we present upper bounds $\phi(\beta)$ and $\psi(\beta)$ on the expected approximation factor of algorithms for, respectively, the minimum dominating set and vertex cover problems in power-law graphs. In our analysis we use a generalized random graph model with expected power-law degree distribution. Let G be a graph with n vertices, V_1 the set of vertices of degree one in G, and $N(V_1)$ the neighborhood of V_1. We show that the combination of a preprocessing step on $N(V_1) \cup V_1$ and the execution of an approximation algorithm in the graph induced by $V \setminus \{N(V_1) \cup V_1\}$ leads to values for $\phi(\beta)$ and $\psi(\beta)$ that do not depend on n and outperforms previous results in literature. More specifically, we show that in the minimum dominating set problem, $\phi(\beta)$ is asymptotically at most 9.14 for $2.1 \leq \beta \leq 2.729$, and 3.68 for $2.729 < \beta < 4$, tighter bounds than the ones of Gast et al. (2015). For the vertex cover problem, we show that $\psi(\beta)$ is asymptotically strictly smaller than 2 for $2 < \beta < 4$, outperforming the bounds of Gast and Hauptmann (2014) and Vignatti and da Silva (2016).

1. Introduction

Empirical studies from the late 1990's and early 2000's [12, 2, 25, 6, 26, 24, 19, 35, 11] pointed out that a number of large real-world networks – also commonly called *complex networks* – from social, biological, and technological applications follow a *power law* on their vertex degree distribution. We can informally describe a power law as a function that decreases in the vertex degree i as i grows large for a fixed exponent $\beta > 0$ and a proportionality constant α, i.e. $f(i) = \alpha i^{-\beta}$. Random graph models for such complex networks are referred as *power-law graphs*. There is evidence that optimization problems might be easier for power-law graphs than for graphs in general [11, 32, 18, 9, 10]. More precisely, if one assumes that the input graph is drawn from a distribution where the expected degree distribution follows a power law, then several problems admit approximation algorithms with expected factors that may not be achievable for general graphs [36, 15, 16, 17].

Random graph models with arbitrary degree distributions have been studied since at least the late 1970's [3, 38, 4, 29, 30, 7, 8, 5]. In this paper we use the *generalized random graph* (GRG) model, introduced by Britton et al. [5], which is a generalization of the well-known Erdős–Rényi random graph model, where weights are assigned to the vertices of the graph. These weights are used for obtaining an arbitrary expected distribution for the vertex degrees. One advantage of this model is that the edges of the graph are created independently. In order to have an expected power-law distribution, we use the sequence of weights given by the formula described in the work of Aiello et al. [1]. The authors propose a random graph model known as ACL(α, β), which is also a model for power-law graphs, but it does not have the convenience of having independent edge probabilities.

We refer to the random graph model used in this paper as GRG(α, β) (the precise definitions are given in Section 2). We note that the well-known Chung–Lu model [7, 8] also uses a sequence of weights for the vertices, so that the expected degree of each vertex corresponds to its weight. In the work of Vignatti and da Silva [36], the authors show that the edge probabilities of the Chung–Lu model and the GRG(α, β) are asymptotically the same for the particular degree sequence that we are using in this work. As a consequence, every result present in this paper also holds for the Chung–Lu model.

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The main result we prove in this work is a lower bound for the expected size of the neighborhood of vertices of degree one. As a consequence, we obtain tighter bounds for the approximability of both the minimum dominating set and the vertex cover problems, improving the previous results from Gast et al. [17] and Vignatti and da Silva [36], respectively. The minimum dominating set (MDS) problem consists of finding the minimum set of vertices $D \subseteq V$ in a graph $G = (V, E)$ such that each $v \in V$ is either in D or has at least one neighbor in D. The minimum vertex cover (MVC) problem corresponds to finding the minimum set $C \subseteq V$ such that each $e \in E$ has at least one endpoint in C [14]. Both problems are $\mathcal{NP}$-Hard [14] and have applications in a variety of contexts and scenarios [37, 39, 40, 31, 20, 21, 28]. In fact, Ferrante et al. [13] showed that these problems remain $\mathcal{NP}$-Hard for the (deterministic) class of graphs respecting the degree distribution given by the formula described in the ACL(α, β) model [1].

The minimum dominating set problem is conjectured not to admit a polynomial time approximation algorithm with a strictly sublogarithmic factor unless $\mathcal{P} = \mathcal{NP}$ [33]. Similarly, the vertex cover problem is conjectured not to admit a polynomial time approximation algorithm with a factor smaller than 2 [23]. However, when restricted to power-law graphs, both barriers can be overtaken [15, 17, 36]. An approximation factor of $\mathcal{O}(\log n)$ can be achieved for the MDS problem using an approximation algorithm for graphs in general. Gast et al. [17] showed that the expected factor of approximation for this algorithm is constant when the input graph is a random sample from the ACL(α, β) model. In this paper we use the GRG(α, β) to show that for $2 < \beta \leq 2.52$ and $2.729 < \beta < 2.85$ the expected approximation factor is significantly smaller than the one obtained in [17]. We note that, in many power-law graphs that model practical applications, β falls between 2 and 3 [6, 22, 27, 34]. Additionally we show that our results also imply a significantly better expected approximation factor for the MVC for graphs in the GRG(α, β) model, for $2 < \beta < 4$, where this factor is near 1 as β gets closer to 4. It is important to highlight, though, that our bounds for the MDS cannot be directly compared with the ones in [15, 17] since the random graph models are not exactly the same.

At the center of our analysis for both the MDS and MVC problems there is a proof of a lower bound for the expected size of the neighborhood of the vertices with degree one. We use this lower bound to estimate the optimal solution obtained by an approximation algorithm together with a simple preprocessing step. Following the previous approaches of [15, 17, 36], the idea is that the neighborhood of degree one vertices is included in the optimal solution – this corresponds to a large portion of the vertices – and an approximation algorithm is used in the remaining part of the graph. The expected approximation factors for the MDS and MVC problems, respectively denoted by $\phi(\beta)$ and $\psi(\beta)$, corresponds to

$$\phi(\beta) \lesssim \frac{\zeta(\beta) + \mathrm{Li}_{\beta-1}(1/e)\left(\frac{\mathrm{Li}_{\beta-1}(1/e)}{2\zeta(\beta-1)} - 1\right)}{\zeta(\beta)\rho(\beta) - \frac{(\mathrm{Li}_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}}$$

and

$$\psi(\beta) \lesssim 2 - \left(\frac{\rho(\beta)}{1 - \frac{\mathrm{Li}_{\beta}(1/e)}{\zeta(\beta)} - \frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta)}}\right),$$

where $\rho(\beta) \approx 1 - \frac{\mathrm{Li}_{\beta}\left(\left(\frac{1}{e}\right)^{\frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta-1)}}\right)}{\zeta(\beta)}$, for $2 < \beta < 4$. The symbols "$\approx$" and "$\lesssim$" denote asymptotic approximations for, respectively, equality and upper bound (see Section 2). The upper bounds of $\phi(\beta)$ and $\psi(\beta)$ can be better understood from Figures 1 and 2. As far as we know, the expected approximation factors obtained for both problems are the best for power-law graphs.

This paper is organized as follows: in Section 2 we provide the definitions of our random graph model; in Section 3 we present the crux of our analysis, which is a lower bound for the neighborhood of the degree one vertices; in Section 4 we show our strategy for dealing with the approximability of the MDS problem; Section 5 describes our new results for approximability of the MVC problem, and Section 6 presents the concluding remarks and directions for future work.

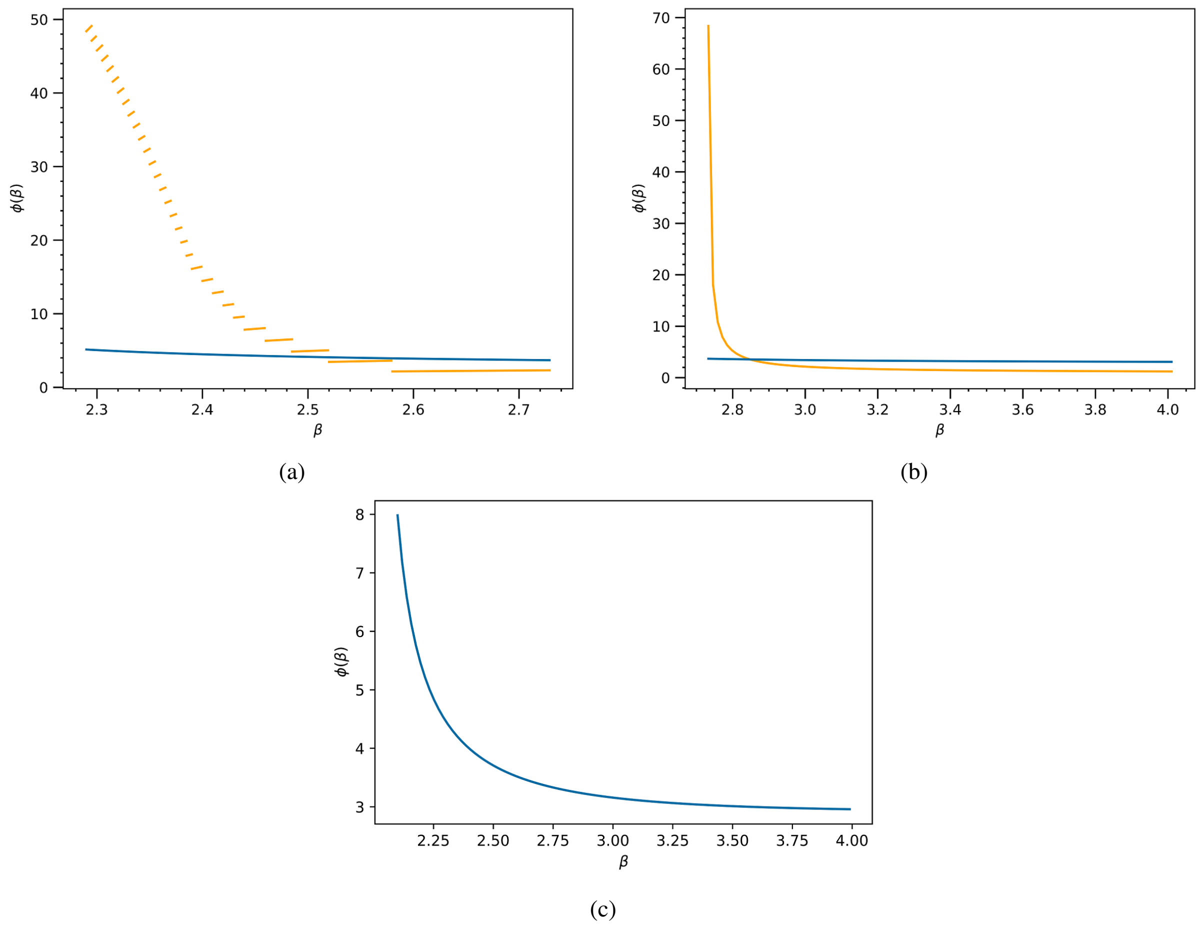

Figure 1: In (c), the graph of our approximation factor for the minimum dominating set problem $2 < \beta < 4$. In (a) and (b), we compare our bound (darker blue line) with the results of Gast et al. [17] (lighter orange line).

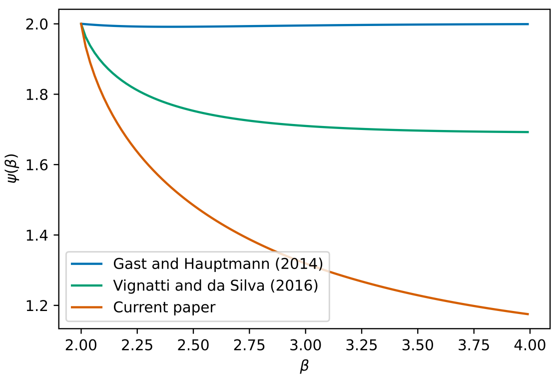

Figure 2: Comparison of the expected approximation factor between our work and the results of Gast et al. [15] and Vignatti and da Silva [36], for $2 < \beta < 4$.

2. Preliminaries

Throughout this paper, we use $\approx$ to denote an *asymptotic approximation*, i.e. given functions $f(n)$ and $g(n)$, then $f(n) \approx g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. We also use $\lesssim$ and $\gtrsim$, respectively, to denote an *asymptotic upper* and *lower bound approximation*. Formally, we have that $f(n) \lesssim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq 1$, and $f(n) \gtrsim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} \geq 1$. It is worth mentioning that the lower and upper asymptotic approximations that are used here are stronger than the Ω and $\mathcal{O}$ asymptotic notations.

For the next definitions and throughout the results of this paper, we denote $\zeta(\beta) = \sum_{j=1}^{\infty} \frac{1}{j^\beta}$ the Riemann zeta function and $\mathrm{Li}_\beta(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^\beta}$ the polylogarithmic function.

Let $G = (V, E)$ be a random graph with $n = |V|$ and $m = |E|$. Consider the vertex set $V = \{1, 2, \dots, |V|\}$. In this work we use the GRG model proposed by Britton et al. [5], where there is a weight w_v associated to each vertex $v \in V$. We denote W_k the set of vertices having weight k, i.e. $W_k = \{v \in V \mid w_v = k\}$. Let w be a vector with entries $w_1, ..., w_{|V|}$. In the GRG model, every edge ij is created independently at random with probability $\Pr(ij \in E) = \frac{w_i w_j}{\ell_n + w_i w_j}$, where $\ell_n = \sum_{v\in V} w_v$. In the literature p_{ij} usually refers to the probability of an edge connecting vertex i and vertex j. For the sake of convenience, however, we refer to p_{ij} as the probability of a vertex having weight i connects to a vertex having weight j.

Naturally, the vertex degrees depends on w, so we set the weights in such vector using similar principles of the ones in Aiello et al. [1] to create a power-law random graph with exponent $\beta > 2$. Consider $y_j = \left\lfloor \frac{e^\alpha}{j^\beta} \right\rfloor$, for each $j = 1, \dots, \Delta$, where $\Delta = \lfloor e^{\alpha/\beta} \rfloor$ and $\alpha = \ln\left(\frac{|V|}{\zeta(\beta)}\right)$. On the ACL$(\alpha, \beta)$ model, there are y_j vertices of fixed degree j. Similarly, in our model, we assign weight j to y_j vertices. We denote by GRG(α, β) a GRG random graph having such distribution on its vertex degrees.

Note that, from the definition of α, we have $|V| = e^\alpha \zeta(\beta)$. Aiello et al. [1] observe that we can ignore rounding in the values of y_j and Δ. However, some extra care has to be taken in the values of y_j in the ACL(α, β) model, since the vertex degrees sequence must be a graphic sequence. In the GRG(α, β) model we do not need such restriction since y_j is associated to the weights and not to the degrees.

Using the y's values defined above, note that

$$\ell_n = \sum_{v\in V} w_v = \sum_{j=1}^{\Delta} j \cdot y_j \approx \sum_{j=1}^{\Delta} j \cdot \frac{e^\alpha}{j^\beta} \approx e^\alpha \zeta(\beta - 1),$$

and hence, and edge connecting a vertex of degree i with a vertex of degree j is created independently at random with probability $p_{ij} = \frac{ij}{e^\alpha \zeta(\beta-1)+ij}$. In Lemma 2.1 in [36], the authors show that $p_{ij} \approx \frac{ij}{e^\alpha \zeta(\beta-1)}$. On the other hand, using the y's values on the Chung–Lu model [7, 8], we have $p_{ij} = \frac{ij}{e^\alpha \zeta(\beta-1)}$. Thus, we conclude that GRG(α, β) and Chung–Lu models are asymptotically equivalent for the power-law weight distribution that we use here, and all results in this paper hold in the Chung–Lu model.

We use the notation $u \to v$ to refer to the event where the vertex u is adjacent to v in the resulting graph G. The degree of $v \in V$ is denoted by $d(v)$ and we denote V_k the set of vertices of degree k.

Let $V^- = V \setminus \{V_0 \cup V_1\}$. For $S \subseteq V$, denote $G[S]$ the graph induced by S and denote $N(S)$ the neighborhood of S in G, i.e. the set of vertices that are adjacent to a vertex of S. The set $N(V_1)$ denotes the neighborhood of V_1 in G and it can be expressed as $N(V_1) = N(V_1)^- \cup N(V_1)^{(1)}$, where $N(V_1)^-$ corresponds to the set of vertices in $N(V_1)$ that have degree greater than one and $N(V_1)^{(1)}$ are vertices of $N(V_1)$ that have degree equal to one.

Lemma 1. *(see [36], Lemma 3.1) Let $q_{ik} = 1 - p_{ik}$. Then*

$$\prod_{k=1}^{\Delta} q_{ik}^{|W_k|} \approx \frac{1}{e^i}.$$

Lemma 2. *(see [36], Lemma 3.2)*

$$\Pr(v \in W_i) = \frac{(e^\alpha / i^\beta)}{e^\alpha \zeta(\beta)} = \frac{1}{i^\beta \zeta(\beta)}.$$

Lemma 3. *(see [36], Lemma 3.3)*

$$\Pr(v \in V_0 \mid v \in W_i) \approx \frac{1}{e^i}.$$

Lemma 4. *(see [36], Lemmas 3.5 and 3.6)*

$$\Pr(v \in V_0) \approx \frac{Li_\beta(1/e)}{\zeta(\beta)} \quad \textit{and} \quad \Pr(v \in V_1) \approx \frac{Li_{\beta-1}(1/e)}{\zeta(\beta)}.$$

Lemma 5. *Let $q_{jk} = 1 - p_{jk}$, where $p_{jk} = \frac{jk}{e^\alpha \zeta(\beta-1)}$. Then*

$$\prod_{l=1}^{\Delta} (q_{kl} q_{il})^{|W_l|} \approx \frac{1}{e^{i+k}}.$$

Proof. Trivially from Lemma 1. □

3. Technical lemmas

The main result of this section is the expected value of $|N(V_1)|$ and its corresponding parts, i.e. $|N(V_1)^-|$ and $|N(V_1)^{(1)}|$. We show these results in Lemmas 9, 13, and 14. The size of these sets are crucial for the approximation algorithms presented in Sections 4 and 5. For both algorithms we can run a preprocessing step in the set of vertices in $N(V_1)^-$ and $N(V_1)^{(1)}$. We observe that the vertices in $N(V_1)^{(1)}$ are all in V_1 and each edge between vertices from this set corresponds to an isolated edge.

A first observation is that we are interested in estimating the size of large sets, such as V_1 and $N(V_1)$. These sets grow asymptotically with the size of the graph. On the other hand, for large graphs, probabilities of events related to one particular vertex or one particular edge are asymptotically negligible, as shown in Lemma 6. We combine these two facts in Lemmas 7 and 8 in order to show that for a given vertex v, adjacent to a given vertex w, the asymptotic probability of the event $d(v) = 1$ is the same of the event $d(v) = 0$ in the graph induced by $V \setminus \{w\}$.

Lemma 6. *Consider $j, k \in \{1, \dots, e^{\alpha/\beta}\}$ and $q_{jk} = 1 - p_{jk}$, where*

$$p_{jk} = \frac{jk}{e^\alpha \zeta(\beta - 1)}.$$

Then, $q_{jk} \approx 1$.

Proof. Using the fact $\beta > 2$,

$$\lim_{\alpha\to\infty} \frac{jk}{e^\alpha \zeta(\beta-1)} \le \frac{1}{\zeta(\beta-1)} \lim_{\alpha\to\infty} \frac{e^{\frac{\alpha}{\beta}} e^{\frac{\alpha}{\beta}}}{e^\alpha} = \frac{1}{\zeta(\beta-1)} \lim_{\alpha\to\infty} e^{\alpha\left(\frac{2}{\beta}-1\right)} = 0.$$

□

Lemma 7. *Consider $(u, w) \in V^2$ such that $u \in W_j$ and $w \in W_i$. Then*

$$\Pr(u \in V_1 \mid u \in W_j \textit{ and } w \in W_i \textit{ and } w \to u) \approx \Pr(u \in V_0 \mid u \in W_j).$$

Proof. Let X_v be the binary random variable associated to vertex u such that $X_v = 1$ if $u \to v$, $X_v = 0$ otherwise. Note that these binary random variables are mutually independent, since edges are independently generated in our random graph model. We now compute the probability of u not being adjacent to any other vertex in V except w. That is,

$$\begin{aligned}
&\Pr(u \in V_1 \mid u \in W_j \text{ and } w \in W_i \text{ and } w \to u)\\
&\quad = \Pr\left(\bigcap_{\substack{v\in V\\ v\neq u\neq w}} X_v = 0 \middle| u \in W_j \text{ and } w \in W_i \text{ and } w \to u \right)\\
&\quad = \prod_{\substack{v\in V\\ v\neq u\neq w}} \Pr(X_v = 0 \mid u \in W_j \text{ and } w \in W_i \text{ and } w \to u)\\
&\quad = \frac{1}{q_{ij} q_{jj}} \prod_{k=1}^{\Delta} \prod_{v\in W_k} q_{jk} = \frac{1}{q_{ij} q_{jj}} \prod_{k=1}^{\Delta} q_{jk}^{|W_k|} \approx \frac{1}{e^j} \frac{1}{q_{ij} q_{jj}} \approx \frac{1}{e^j} \approx \Pr(u \in V_0 \mid u \in W_j),
\end{aligned}$$

where the approximations follow from Lemmas 1, 3, and 6. $\square$

Lemma 8. *Consider $(u, v) \in V^2$ such that $u \in W_i$ and $v \in W_j$. Then*

$$\Pr(u \in V_1 \textit{ and } v \in V_1 \mid u \to v \textit{ and } u \in W_i \textit{ and } v \in W_j) \approx \Pr(u \in V_0 \mid u \in W_i) \cdot \Pr(v \in V_0 \mid v \in W_j).$$

Proof. Consider the random variable X_{zu} with respect to u, defined for each $z \in V$, such that $X_{zu} = 1$ if $z \to u$ (and $X_{zu} = 0$ otherwise). The random variable X_{zv} is defined analogously to X_{zu}.

Note that each X_{zu} (and X_{zv}) are mutually independent, since edges are independently generated in our random graph model. We now compute the probability of u not being adjacent to any other vertex in V except v (and vice-versa for v). Then

$$\begin{aligned}
&\Pr(u \in V_1 \text{ and } v \in V_1 \mid u \to v \text{ and } u \in W_i \text{ and } v \in W_j)\\
&= \Pr\left(\bigcap_{\substack{z\in V\\ z\neq u\neq v}} (X_{zu} = 0) \text{ and } \bigcap_{\substack{z\in V\\ z\neq u\neq v}} (X_{zv} = 0) \middle| u \in W_i \text{ and } v \in W_j \text{ and } u \to v \right)\\
&= \Pr\left(\bigcap_{\substack{z\in V\\ z\neq u\neq v}} (X_{zu} = 0) \middle| u \in W_i \text{ and } v \in W_j \text{ and } u \to v \right) \cdot \Pr\left(\bigcap_{\substack{z\in V\\ z\neq u\neq v}} (X_{zv} = 0) \middle| u \in W_i \text{ and } v \in W_j \text{ and } u \to v \right)\\
&= \prod_{\substack{z\in V\\ z\neq u\neq v}} \Pr(X_{zu} = 0 \mid u \in W_i \text{ and } v \in W_j \text{ and } u \to v) \cdot \prod_{\substack{z\in V\\ z\neq u\neq v}} \Pr(X_{zv} = 0 \mid u \in W_i \text{ and } v \in W_j \text{ and } u \to v)\\
&\approx \frac{1}{e^i} \frac{1}{q_{ii}q_{ij}} \frac{1}{e^j} \frac{1}{q_{ij}q_{jj}} \approx \frac{1}{e^j}\frac{1}{e^i} \approx \Pr(u \in V_0 \mid u \in W_i) \Pr(v \in V_0 \mid v \in W_j),
\end{aligned}$$

where the second and third equations follow since the events are mutually independent, and the approximations follow from Lemmas 1, 3, and 6. $\square$

Lemma 9.

$$\mathrm{E}[|N(V_1)^{(1)}|] \approx \frac{e^\alpha (Li_{\beta-1}(1/e))^2}{\zeta(\beta-1)}.$$

Proof. Consider the binary random variable X_{uv} defined as follows:

$$X_{uv} = \begin{cases} 1, & \text{if } u \in V_1 \text{ and } v \in V_1 \text{ and } u \to v \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned}
&\Pr(u \in V_1 \text{ and } v \in V_1 \text{ and } u \to v)\\
&= \sum_{i=1}^{\Delta} \Pr(u \in V_1 \text{ and } v \in V_1 \text{ and } u \to v \mid u \in W_i) \Pr(u \in W_i)\\
&= \sum_{i=1}^{\Delta}\sum_{j=1}^{\Delta} \Pr(u \in V_1 \text{ and } v \in V_1 \text{ and } u \to v \mid u \in W_i \text{ and } v \in W_j) \Pr(v \in W_j) \Pr(u \in W_i).
\end{aligned}$$

For given events A, B, and C, by the definition of conditional probability we have that

$$\Pr(A \text{ and } B \mid C) = \frac{\Pr(A \mid B \text{ and } C)\Pr(B \mid C)\Pr(C)}{\Pr(C)} = \Pr(A \mid B \text{ and } C)\Pr(B \mid C).$$

Therefore,

$$
\begin{aligned}
&\Pr(u \in V_1 \text{ and } v \in V_1 \text{ and } u \to v \mid u \in W_i \text{ and } v \in W_j) \\
=&\Pr(u \in V_1 \text{ and } v \in V_1 \mid u \to v \text{ and } u \in W_i \text{ and } v \in W_j) \Pr(u \to v \mid u \in W_i \text{ and } v \in W_j) \\
\approx&\Pr(u \in V_0 \mid u \in W_i) \cdot \Pr(v \in V_0 \mid v \in W_j) \Pr(u \to v \mid u \in W_i \text{ and } v \in W_j)
\end{aligned}
$$

where the approximation is given by Lemma 8. By Lemmas 2 and 3, we have

$$
\begin{aligned}
\mathrm{E}[|N(V_1)^{(1)}|] &= \sum_{(u,v)\in V^2} \Pr(X_{uv} = 1) \\
&= \sum_{(u,v)\in V^2} \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} (\Pr(X_{uv} = 1 \mid u \in W_i \text{ and } v \in W_j) \cdot \Pr(u \in W_i) \cdot \Pr(v \in W_j)) \\
&\approx \sum_{(u,v)\in V^2} \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} \big(\Pr(u \in V_0 \mid u \in W_i) \cdot \Pr(v \in V_0 \mid v \in W_j) \\
&\cdot \Pr(u \to v \mid u \in W_i \text{ and } v \in W_j) \Pr(u \in W_i) \Pr(v \in W_j)\big) \\
&\approx \sum_{(u,v)\in V^2} \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} \frac{1}{e^{i+j}} \frac{ij}{e^{\alpha}\zeta(\beta-1)} \frac{1}{(ij)^{\beta}\zeta(\beta)^2} \\
&= \frac{1}{e^{\alpha}\zeta(\beta-1)\zeta(\beta)^2} \sum_{(u,v)\in V^2} \sum_{i=1}^{\Delta} \frac{i}{e^i i^{\beta}} \sum_{j=1}^{\Delta} \frac{j}{e^j j^{\beta}} \\
&\approx \frac{e^{2\alpha}\zeta(\beta)^2(\mathrm{Li}_{\beta-1}(1/e))^2}{e^{\alpha}\zeta(\beta-1)\zeta(\beta)^2} = \frac{e^{\alpha}(\mathrm{Li}_{\beta-1}(1/e))^2}{\zeta(\beta-1)}.
\end{aligned}
$$

□

Given a fixed vertex v of weight j and a set of vertices $Y \subseteq V$ adjacent to v, we show in Lemma 10 that all events of the type "y is adjacent only to v", $y \in Y$, are approximately mutually independent.

Lemma 10. *For fixed $v \in V$ with weight j, for any $u \in V$ with weight i, and for a subset $S \subseteq V$, such that $u \notin S$,*

$$
\Pr\left(v \to u \textit{ and } u \in V_1 \,\middle|\, \bigcap_{y\in S} (v \to y \textit{ and } y \in V_1) \right) \approx \Pr(v \to u \textit{ and } u \in V_1).
$$

Proof. We have that

$$
\begin{aligned}
&\Pr\left(u \in V_1 \text{ and } v \to u \,\middle|\, \bigcap_{y\in S} (v \to y \text{ and } y \in V_1) \right) \\
=&\frac{\Pr\left(u \in V_1 \text{ and } v \to u \text{ and } \bigcap_{y\in S} v \to y \text{ and } \bigcap_{y\in S} y \in V_1\right)}{\Pr\left(\bigcap_{y\in S} v \to y \text{ and } \bigcap_{y\in S} y \in V_1\right)} \\
=&\frac{\Pr\left(u \in V_1 \text{ and } \bigcap_{y\in S} y \in V_1 \,\middle|\, v \to u \text{ and } \bigcap_{y\in S} v \to y\right) \Pr\left(v \to u \,\middle|\, \bigcap_{y\in S} v \to y\right)}{\Pr\left(\bigcap_{y\in S} y \in V_1 \,\middle|\, \bigcap_{y\in S} v \to y\right)}
\end{aligned}
$$

by the fact that

$$
\Pr(A \text{ and } B \mid C \text{ and } D) = \frac{\Pr(A \text{ and } B \text{ and } C \text{ and } D)}{\Pr(C \text{ and } D)}
$$

$$= \frac{\Pr(A \text{ and } C \mid B \text{ and } D)\Pr(B \mid D)\Pr(D)}{\Pr(C \mid D)\Pr(D)} = \frac{\Pr(A \text{ and } C \mid B \text{ and } D)\Pr(B \mid D)}{\Pr(C \mid D)}.$$

Consider a vertex w with weight k. Let X_{uw} be the binary random variable having $X_{uw} = 1$ if $w \to u$ (and $X_{uw} = 0$ otherwise). The set of events $w \to u$, for each $w \in V$, is mutually independent. Then

$$\Pr\left(u \in V_1 \text{ and } \bigcap_{y \in S} y \in V_1 \middle| v \to u \text{ and } \bigcap_{y \in S} v \to y\right)$$

$$= \Pr\left(\bigcap_{y \in S} X_{yu} = 0 \text{ and } \bigcap_{y \in S} X_{uy} = 0 \ \text{ and } \bigcap_{y \in S}\ \bigcap_{y' \in S : y \neq y'} (X_{yy'} = 0 \text{ and } X_{y'y} = 0)\right.$$

$$\left.\text{and } \bigcap_{w \in \{V \setminus S\} : w \neq u \neq v} X_{wu} = 0 \text{ and } \bigcap_{y \in S}\ \bigcap_{w \in \{V \setminus S\} : w \neq u \neq v} X_{wy} = 0\right)$$

$$= \prod_{y \in S} \Pr(X_{yu} = 0 \text{ and } X_{uy} = 0) \prod_{y \in S} \prod_{\substack{y' \in S \\ y \neq y'}} \Pr(X_{yy'} = 0 \text{ and } X_{y'y} = 0)$$

$$\prod_{\substack{w \in V \setminus S \\ w \neq u \neq v}} \Pr(X_{wu} = 0) \prod_{y \in S} \prod_{\substack{w \in V \setminus S \\ w \neq u \neq v}} \Pr(X_{wy} = 0).$$

We have that

$$\Pr\left(\bigcap_{y \in S} y \in V_1 \middle| \bigcap_{y \in S} v \to y\right)$$

$$= \Pr\left(\bigcap_{y \in S}\ \bigcap_{y' \in S : y \neq y'} (X_{yy'} = 0 \text{ and } X_{y'y} = 0) \text{ and } \bigcap_{y \in S}\ \bigcap_{w \in \{V \setminus S\} : w \neq u \neq v} X_{wy} = 0 \text{ and } \bigcap_{y \in S} X_{uy} = 0\right)$$

$$= \prod_{y \in S} \prod_{\substack{y' \in S \\ y \neq y'}} \Pr(X_{yy'} = 0 \text{ and } X_{y'y} = 0) \prod_{y \in S} \prod_{\substack{w \in V \setminus S \\ w \neq u \neq v}} \Pr(X_{wy} = 0) \prod_{y \in S} \Pr(X_{uy} = 0)$$

$$= \prod_{y \in S} \prod_{\substack{y' \in S \\ y \neq y'}} \Pr(X_{yy'} = 0 \text{ and } X_{y'y} = 0) \prod_{y \in S} \prod_{\substack{w \in V \setminus S \\ w \neq u \neq v}} \Pr(X_{wy} = 0) \cdot \prod_{y \in S} \Pr(X_{uy} = 0) \prod_{y \in S} \Pr(X_{uy} = 0 \text{ and } X_{yu} = 0).$$

The last equality comes from the fact that

$$\prod_{y \in S} \Pr(X_{uy} = 0 \text{ and } X_{yu} = 0) = \prod_{y \in S} \Pr(X_{yu} = 0 \mid X_{uy} = 0) \Pr(X_{uy} = 0)$$

$$= \prod_{y \in S} \Pr(X_{yu} = 0).$$

In addition, the event $v \to u$ is independent from $\bigcap_{y \in S} v \to y$, and hence,

$$\Pr\left(v \to u \middle| \bigcap_{y \in S} v \to y\right) = \Pr(v \to u) \approx \frac{ij}{e^{\alpha} \zeta(\beta - 1)}.$$

Therefore,

$$
\begin{aligned}
&\Pr\left(v \to u \text{ and } u \in V_1 \middle| \bigcap_{y\in S} (v \to y \text{ and } y \in V_1) \right) \\
=&\frac{ij}{e^{\alpha}\zeta(\beta-1)} \prod_{y\in S} \Pr(X_{yu}=0) \prod_{\substack{w\in V\setminus S \\ w\neq u\neq v}} \Pr(X_{wu}=0) \\
=&\frac{ij}{e^{\alpha}\zeta(\beta-1)} \prod_{\substack{w\in V \\ w\neq u\neq v}} \Pr(X_{wu}=0) \\
\approx&\frac{ij}{e^{\alpha}\zeta(\beta-1)} \frac{1}{e^i} \frac{1}{q_{ii}q_{ij}} \approx \frac{ij}{e^i e^{\alpha}\zeta(\beta-1)}
\end{aligned}
$$

where the approximations in the last line come from Lemmas 5 and 6.
By Lemma 7, this corresponds to

$$\Pr(u \in V_1 \text{ and } v \to u) = \Pr(u \in V_1 \mid u \to v) \Pr(u \to v).$$

This concludes the proof. □

Corollary 1. *For fixed $v \in V$ with weight j and for any $u \in V$ with weight i, the events $v \to u \wedge u \in V_1$ are approximately mutually independent.*

For the lemmas and theorems below, we denote by $v \longrightarrow S$ the event of the vertex v be connected to the set $S \subseteq V$.

Lemma 11.

$$\Pr(v \longrightarrow V_1 \mid v \in W_j) \gtrsim 1 - \left(\frac{1}{e}\right)^{\frac{jLi_{\beta-1}(1/e)}{\zeta(\beta-1)}}.$$

Proof. Let X_u be the binary random variable associated to $u \in W_i$, for $1 \le i \le \Delta$, such that $X_u = 1$ if $v \to u$ and $u \in V_1$ (and $X_u = 0$ otherwise). From De Morgan's law, from Corollary 1, and Lemma 10, and the fact that $\left(1-\frac{a}{x}\right)^x \le \left(\frac{1}{e}\right)^a$, we have

$$
\begin{aligned}
\Pr(v \longrightarrow V_1 \mid v \in W_j) =&\Pr\left(\bigcup_{u\in V} (v \to u \text{ and } u \in V_1) \middle| v \in W_j \right) \\
=&1 - \Pr\left(\bigcap_{u\in V} (v \nrightarrow u \text{ or } u \notin V_1) \middle| v \in W_j \right) \\
=&1 - \Pr\left(\bigcap_{i=1}^{\Delta} \bigcap_{u\in W_i} (X_u = 0) \middle| v \in W_j \right) \\
\approx&1 - \prod_{i=1}^{\Delta} \prod_{u\in W_i} \left(1 - \frac{ij}{e^i e^{\alpha}\zeta(\beta-1)}\right) \\
=&1 - \prod_{i=1}^{\Delta} \left(1 - \frac{ij}{e^i e^{\alpha}\zeta(\beta-1)}\right)^{e^{\alpha}/i^{\beta}} \\
\gtrsim&1 - \prod_{i=1}^{\Delta} \left(\frac{1}{e}\right)^{\frac{ij}{e^i\zeta(\beta-1)i^{\beta}}} = 1 - \left(\frac{1}{e}\right)^{\sum_{i=1}^{\Delta} \frac{ij}{e^i\zeta(\beta-1)i^{\beta}}} \\
\approx&1 - \left(\frac{1}{e}\right)^{\frac{j\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta-1)}}.
\end{aligned}
$$

□

Let $\rho(\beta) \approx 1 - \frac{\mathrm{Li}_\beta\left(\left(\frac{1}{e}\right)^{\frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta-1)}}\right)}{\zeta(\beta)}$.

Lemma 12.

$$\Pr(v \longrightarrow V_1) \gtrsim \rho(\beta).$$

Proof. By Lemmas 2 and 11,

$$\begin{aligned}\Pr(v \longrightarrow V_1) &= \sum_{j=1}^{\Delta} \Pr(v \longrightarrow V_1 \mid v \in W_j) \Pr(v \in W_j) \\ &\gtrsim \sum_{j=1}^{\Delta} \left(1 - \left(\frac{1}{e}\right)^{\frac{j\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta-1)}}\right) \frac{1}{j^\beta \zeta(\beta)} \approx 1 - \frac{\mathrm{Li}_\beta\left(\left(\frac{1}{e}\right)^{\frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta-1)}}\right)}{\zeta(\beta)}.\end{aligned}$$

□

Lemma 13.

$$\mathrm{E}[|N(V_1)|] \gtrsim e^\alpha \zeta(\beta)\rho(\beta).$$

Proof. Let X_v be the binary random variable associated to $v \in V$ such that $X_v = 1$ if $v \longrightarrow V_1$ (and $X_v = 0$ otherwise). Then by Lemma 12,

$$\mathrm{E}[|N(V_1)|] = \sum_{v \in V} \Pr(v \longrightarrow V_1) \gtrsim e^\alpha \zeta(\beta)\rho(\beta).$$ □

Lemma 14.

$$\mathrm{E}[|N(V_1)^-|] \gtrsim e^\alpha \left(\zeta(\beta)\rho(\beta) - \frac{(Li_{\beta-1}(1/e))^2}{\zeta(\beta-1)} \right).$$

Proof. Directly from $\mathrm{E}[|N(V_1)|] = \mathrm{E}[|N(V_1)^-|] + \mathrm{E}[|N(V_1)^{(1)}|]$, and Lemmas 9 and 13. □

4. Approximation algorithm for the minimum dominating set problem

The strategy that we use for finding an approximation is similar to the one of Gast and Hauptmann (2015) [17]. We start with a preprocessing step where we include every vertex of $N(V_1)^-$ and half of the vertices of $N(V_1)^{(1)}$ in the solution. Then we apply an approximation algorithm in the graph induced by $V \setminus \{N(V_1) \cup V_1\}$. Consider the set $N(V_1)^{(1)'} \subseteq N(V_1)$, where $|N(V_1)^{(1)'}| = |N(V_1)^{(1)}|/2$, and denote by R the set $R = V \setminus \{N(V_1)^- \cup V_1\}$. In Lemma 15 we prove that the approximation factor $\phi(\beta)$ for the minimum dominating set problem corresponds to $\frac{r|\mathrm{OPT}(R)|+|N(V_1)^-|+|N(V_1)^{(1)'}|}{|\mathrm{OPT}(R)|+|N(V_1)^-|+|N(V_1)^{(1)'}|}$, where $\mathrm{OPT}(R)$ is the optimal dominating set in R. We observe that, as in Lemma 4.1 in [36], this holds for any graph G (i.e. no probabilistic argument is used in the proof). In the next results in this section, with the exception of Lemma 15, we treat the sizes of $\mathrm{OPT}(R)$, $N(V_1)$, and R as expected values of random variables. The bounds for the approximation factor given by Theorem 1 and Corollary 3 are illustrated in Figures 1 and 3, respectively, where we compare our results with the bounds of Gast and Hauptmann (Theorem 4 [17]). Due to the nature of the random graphs the authors use, they obtained two functions for $\phi(\beta)$, defining the appropriated ranges for β in each case.

Lemma 15. *The approximation factor $\phi(\beta)$ for the minimum dominating set problem is at most*

$$\frac{r|OPT(R)| + |N(V_1)^-| + |N(V_1)^{(1)'}|}{|OPT(R)| + |N(V_1)^-| + |N(V_1)^{(1)'}|},$$

where r is the approximation factor of the algorithm applied to set R.

Proof. Consider $V^* = V_1 \cup N(V_1)$. We first prove that the following two conditions hold:

(i) G contains a minimum dominating set D such that $(N(V_1)^- \cup N(V_1)^{(1)'}) \subseteq D$, and

(ii) $\mathrm{OPT}(V^*) = |N(V_1)^-| + |N(V_1)^{(1)'}|$.

For each edge $(x, y) \in E$ such that $x \in V_1$ and $y \in V^-$, either x or y (but not both) must belong to D (otherwise D is not minimum). If $x \in V_1$, then $(D \setminus \{x\}) \cup \{y\}$ is also a minimum dominating set, then, using the same exchange argument, there is a minimum dominating set containing every vertex of $N(V_1)^-$. For each pair of vertices $(x, y) \in V_1$ where $x \to y$, then either x or y (but not both) must belong to D, therefore, half of the vertices from $N(V_1)^{(1)}$ are in D. We denote such set by $N(V_1)^{(1)'}$. So, (i) holds.

From (i), we have that the graph induced by $N(V_1)^- \cup N(V_1)^{(1)'}$ is an optimal solution for $G[V^*]$. Besides, sets $N(V_1)^-$ and $N(V_1)^{(1)}$ are disjoint, and hence, (ii) holds. Now let $\mathrm{OPT}(V)$ denote the size of the optimal solution such that condition (i) holds. From (ii), we have that

$$\begin{aligned}\mathrm{OPT}(V) \leq& |\mathrm{OPT}(R) \cup N(V_1)^-| + |N(V_1)^{(1)'}|\\ =& |\mathrm{OPT}(R)| + |N(V_1)^-| + |N(V_1)^{(1)'}|,\end{aligned}$$

where the last equality comes from the fact that R and $N(V_1)^- = \emptyset$.

Let $\mathrm{OPT}(V)'$ be the size of the solution obtained by the approximation strategy. Then

$$\phi(\beta) \leq \frac{\mathrm{OPT}(V)'}{\mathrm{OPT}(V)} \leq \frac{r|\mathrm{OPT}(R)| + |N(V_1)^-| + |N(V_1)^{(1)'}|}{|\mathrm{OPT}(R)| + |N(V_1)^-| + |N(V_1)^{(1)'}|},$$

where the last inequality comes from the fact that $\frac{za+b}{a+b} \leq \frac{zc+b}{c+b}$ for $z, a, b, c \in \mathbb{R}$, where $z > 1$ and $a \leq c$. □

Corollary 2.

$$\phi(\beta) \leq \frac{r|OPT(R)| + |N(V_1)^-| + |N(V_1)^{(1)'}| + |V_0|}{|OPT(R)| + |N(V_1)^-| + |N(V_1)^{(1)'}| + |V_0|}$$

where r is the approximation factor of the algorithm applied to set R, and V_0 is the set of vertices that have degree 0.

In Theorem 1 we give a constant upper bound for the expected value of $\phi(\beta)$. In the proof of our upper bound we use the next result from [17], adapted to the random graph model we use. The approximation algorithm has an approximation factor given by $\mathcal{O}(\log \Delta)$, where $\Delta = e^{\alpha/\beta}$ is the maximum degree of a vertex in $G[R]$.

Lemma 16. *(see [17], Section 8) For $2 < \beta < 4$,*

$$\begin{aligned}\phi(\beta) = \max&\left\{ \frac{r|OPT(R)| + |N(V_1)^-| + |N(V_1)^{(1)}|/2}{|OPT(R)| + |N(V_1)^-| + |N(V_1)^{(1)}|/2} \;\middle|\; |OPT(R)| \leq |R|,\right.\\ &\left. r = \min\left\{\frac{\alpha}{\beta}, \frac{|R|}{|OPT(R)|}\right\}\right\} \leq \frac{|R| + |N(V_1)^-| + |N(V_1)^{(1)}|/2}{\frac{\beta}{\alpha}|R| + |N(V_1)^-| + |N(V_1)^{(1)}|/2}.\end{aligned}$$

Theorem 1.

$$\phi(\beta) \lesssim \frac{\zeta(\beta) + Li_{\beta-1}(1/e)\left(\frac{Li_{\beta-1}(1/e)}{2\zeta(\beta-1)} - 1\right)}{\zeta(\beta)\rho(\beta) - \frac{(Li_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}},$$

for non-empty $N(V_1)^{(1)}$, for $2 < \beta < 4$.

Proof. From Lemmas 15 and 16, we have that the upper bound for the approximation factor $\phi(\beta)$ corresponds to

$$\phi(\beta) \le \frac{\mathrm{E}[|R|] + \mathrm{E}[|N(V_1)^-|] + \mathrm{E}[|N(V_1)^{(1)}|]/2}{\frac{\beta}{\alpha}\mathrm{E}[|R|] + \mathrm{E}[|N(V_1)^-|] + \mathrm{E}[|N(V_1)^{(1)}|]/2}.$$

By linearity of expectation, $\mathrm{E}[|R|] = |V| - \mathrm{E}[|N(V_1)^-|] - \mathrm{E}[|V_1|]$, since $N(V_1)^-$ and V_1 are disjoint.

Writing $\mathrm{E}[|N(V_1)^-|] \gtrsim e^\alpha a$, $\mathrm{E}[|V_1|] \approx e^\alpha \zeta(\beta) b$, and $\mathrm{E}[|N(V_1)^{(1)}|] \approx e^\alpha c$, where a, b, and c are the constant parts on the expected size of each set, then

$$\phi(\beta) \le \frac{e^\alpha(\zeta(\beta) - a - b) + e^\alpha(a + \frac{c}{2})}{\frac{\beta}{\alpha}e^\alpha(\zeta(\beta) - a - b) + e^\alpha(a + \frac{c}{2})} \le \frac{\zeta(\beta) - b + \frac{c}{2}}{a + \frac{c}{2}}$$

since $\frac{\beta}{\alpha}(\zeta(\beta) - a - b) \ge 0$. The result follows from Lemmas 4, 9, and 14. □

In our analysis, following the same criteria of [17], we did not include vertices of degree 0 in the solution. For the more general case, the approximation factor follows from Corollary 2 and Theorem 1.

Corollary 3. *For non-empty sets $N(V_1)^{(1)}$ and V_0 (set of isolated vertices in G), with $2 < \beta < 4$,*

$$\phi(\beta) \lesssim \frac{\zeta(\beta) + Li_{\beta-1}(1/e)\left(\frac{Li_{\beta-1}(1/e)}{2\zeta(\beta-1)} - 1\right) + Li_\beta(1/e)}{\zeta(\beta)\rho(\beta) - \frac{(Li_{\beta-1}(1/e))^2}{2\zeta(\beta-1)} + Li_\beta(1/e)}.$$

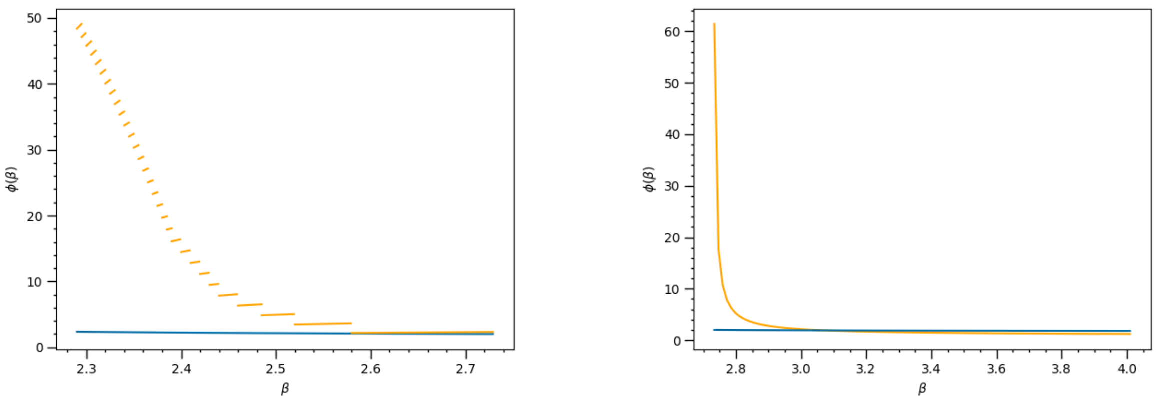

Figure 3: Expected approximation factor given by Corollary 3 for $2 < \beta \le 2.729$ (graph in the left) and $2.729 < \beta < 4$ (graph in the right). The darker line (blue) corresponds to values obtained by our bounds, and the lighter line (orange) corresponds to the expected approximation factor described in Theorem 4 in [17]. In the graph on the left, the function from [17] is not continuous.

5. Approximation algorithm for the vertex cover problem

In this section we show a better factor of approximation for the algorithm for the MVC problem described by Vignatti and da Silva in [36]. The algorithm has an approximation factor strictly smaller than 2 for power-law graphs, what may not be achievable for graphs in general [23]. The approximation factor from [36] is an improvement of a previous result of [15] (although some care should be taken in comparing both results, since the random graph models are not exactly the same, as we have discussed in Section 1). In this section we show that the results obtained in Section 3 imply a significantly better guarantee for the approximation factor for the algorithm of [36]. We illustrate such differences in Figures 2 and 4.

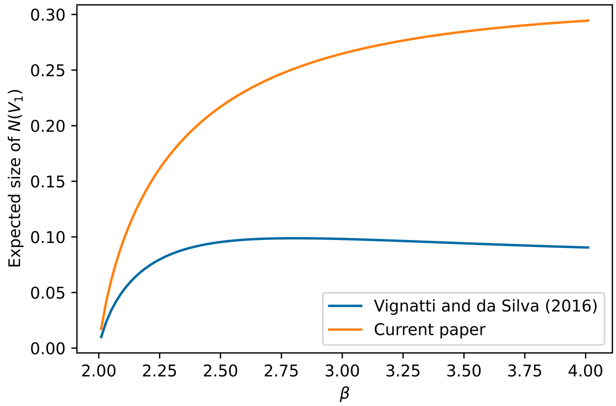

Figure 4: Expected values of $\mathrm{E}[|N(V_1)|]$ as a fraction of V.

The idea is similar, but not identical to the strategy described in Section 4. For the MVC problem we include all vertices of $N(V_1)$ in the solution and then run a 2-approximation algorithm in $V \setminus \{N(V_1) \cup V_1\}$. We state Lemma 17 (the proof is given in [36]) and give the proofs for Lemma 18, Corollary 4, and Theorem 2, although the proofs are similar to the referred paper, for the sake of completeness. Similarly to Section 4, OPT(V), $|N(V_1)|$, and $|V^-|$ are treated as expected values of random variables, except in Lemma 17. For the next lemma, recall that $N(V_1)^{(1)'}$ is the set composed by half of the vertices from $N(V_1)^{(1)}$.

Lemma 17. *(see Lemma 4.1, [36]) The following three conditions hold:*

(i) *G contains a minimum vertex cover C such that $N(V_1)^- \cup N(V_1)^{(1)'} \subseteq C$,*

(ii) *$OPT(V^*) = |N(V_1)^- \cup N(V_1)^{(1)'}|$, and*

(iii) *$OPT(V) = OPT(V^*) + OPT(V \setminus V^*)$,*

where $V^* = V_1 \cup N(V_1)$.

We observe that Lemma 17 (i) is originally stated as "$N(V_1) \subseteq C$ and that there is no vertex of V_1 in C". However, the proof also holds by noting that $N(V_1) = N(V_1)^- \cup N(V_1)^{(1)}$ and that $N(V_1)^- \cup N(V_1)^{(1)'} \subseteq N(V_1)^- \cup N(V_1)^{(1)}$.

Lemma 18. *Let* $\rho(\beta) \approx 1 - \dfrac{Li_\beta\left(\left(\frac{1}{e}\right)^{\frac{Li_{\beta-1}(1/e)}{\zeta(\beta-1)}}\right)}{\zeta(\beta)}$. *Then*

$$\frac{OPT(V^*)}{OPT(V)} \gtrsim \left(\frac{\rho(\beta)}{1 - \frac{Li_\beta(1/e)}{\zeta(\beta)} - \frac{Li_{\beta-1}(1/e)}{\zeta(\beta)} + \frac{(Li_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}}\right).$$

Proof. By Lemma 17 (i), OPT(V) $\leq |V^-| + |N(V_1)^{(1)}|/2$. From Lemma 4,

$$|V^-| \leq |V| \left(1 - \frac{\mathrm{Li}_\beta(1/e)}{\zeta(\beta)} - \frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta)}\right).$$

From Lemma 9,

$$|N(V_1)^{(1)}| \approx \frac{e^\alpha (\mathrm{Li}_{\beta-1}(1/e))^2}{\zeta(\beta-1)} \leq \frac{e^\alpha \zeta(\beta)(\mathrm{Li}_{\beta-1}(1/e))^2}{\zeta(\beta-1)}.$$

By Lemmas 17 (ii) and 13, $\mathrm{OPT}(V^*) = |N(V_1)| \geq |V|\rho(\beta)$. Combining the two bounds, we have

$$\frac{\mathrm{OPT}(V^*)}{\mathrm{OPT}(V)} \gtrsim \left(\frac{\rho(\beta)}{1 - \frac{\mathrm{Li}_\beta(1/e)}{\zeta(\beta)} - \frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta)} + \frac{(\mathrm{Li}_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}} \right).$$

□

Corollary 4.

$$\frac{OPT(V \setminus V^*)}{OPT(V)} \lesssim 1 - \left(\frac{\rho(\beta)}{1 - \frac{Li_\beta(1/e)}{\zeta(\beta)} - \frac{Li_{\beta-1}(1/e)}{\zeta(\beta)} + \frac{(Li_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}} \right).$$

Proof. By Lemma 17 (iii), $\frac{\mathrm{OPT}(V^*)+\mathrm{OPT}(V\setminus V^*)}{\mathrm{OPT}(V)} = 1$. The result holds from Lemma 18. □

Theorem 2. *The expected approximation factor $\psi(\beta)$ for the vertex cover problem corresponds to*

$$\psi(\beta) \lesssim 2 - \left(\frac{\rho(\beta)}{1 - \frac{Li_\beta(1/e)}{\zeta(\beta)} - \frac{Li_{\beta-1}(1/e)}{\zeta(\beta)} + \frac{(Li_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}} \right).$$

Proof. From Lemma 17 we have that an optimal solution has the set $N(V_1)$. Hence, we apply a 2-approximation algorithm in $G[V \setminus V^*]$ and return $C \cup N(V_1)$ as solution, where C is the solution given by the 2-approximation algorithm. Since C and $N(V_1)$ are disjoint, by Lemma 17 ((ii) and (iii)) and Corollary 4,

$$\begin{aligned}
|C \cup N(V_1)| =& |C| + |N(V_1)| \leq 2\mathrm{OPT}(V \setminus V^*) + \mathrm{OPT}(V^*) \\
=& 2\mathrm{OPT}(V \setminus V^*) + \mathrm{OPT}(V) - \mathrm{OPT}(V \setminus V^*) \\
=& \mathrm{OPT}(V \setminus V^*) + \mathrm{OPT}(V) \\
\lesssim& \mathrm{OPT}(V) + \left(1 - \frac{\rho(\beta)}{1 - \frac{\mathrm{Li}_\beta(1/e)}{\zeta(\beta)} - \frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta)} + \frac{(\mathrm{Li}_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}} \right) \mathrm{OPT}(V) \\
=& \left(2 - \frac{\rho(\beta)}{1 - \frac{\mathrm{Li}_\beta(1/e)}{\zeta(\beta)} - \frac{\mathrm{Li}_{\beta-1}(1/e)}{\zeta(\beta)} + \frac{(\mathrm{Li}_{\beta-1}(1/e))^2}{2\zeta(\beta-1)}} \right) \mathrm{OPT}(V).
\end{aligned}$$

□

6. Conclusion

In this paper we present an upper bound $\phi(\beta)$ for the expected approximation factor to the minimum dominating set problem in power-law graphs with $2 < \beta < 4$. We use the generalized random graph model of Britton et al. [5] with expected power-law degree distribution. We show that for $2 < \beta \leq 2.52$ and $2.729 < \beta < 2.85$ the bound is tighter than the one of Gast and Hauptmann [17]. We show that the same techniques can also be applied to the vertex cover problem, improving the previous bound of Vignatti and da Silva [36] for the minimum vertex cover problem. As far as we know, the approximation factors obtained for both problems are the best known factors for power-law graphs.

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