Boolean Algebra

- Boolean algebra (named after G. Boole for his work in 1854) is a mathematical system for specifying and transforming logic functions.

- **Boolean (switching) algebra:** An algebra structure defined on a set of elements, B, together with two binary operators (denoted $+$ and $\cdot$) provided the following **axioms** are satisfied:

  1. Closure with respect to (wrt) $+$ and $\cdot$
  2. Identity elements exist for $+$ and $\cdot$
  3. Commutative wrt $+$ and $\cdot$
  4. Operator $+$ distributes over $\cdot$, and $\cdot$ distributes over $+$.
  5. Every element of B has a complement.
  6. B has at least two distinct elements.

- Purpose: Studying Boolean algebra to specify the design of digital systems!

Algebric Properties: Axioms 1–3 (by Huntington, 1904)

1. **Closure Property**: A set S is **closed** wrt to a binary operator $\bullet$ if and only if (iff) for every $x, y \in S, x \bullet y \in S$.

   - **Axiom 1(a):** B is closed wrt the operator $+$
   - **Axiom 1(b):** B is closed wrt the operator $\cdot$

2. **Identity Element**: A set S is said to have an **identity element** wrt to a particular binary operator $\bullet$ whenever there exists an element $e \in S$ such that for every $x \in S, e \bullet x = x \bullet e = x$.

   - **Axiom 2(a):** B has an identity element wrt $+$, denoted by 0
   - **Axiom 2(b):** B has an identity element wrt $\cdot$, denoted by 1

3. **Commutativity Property**: A binary operator $\bullet$ defined on a set S is said to be **commutative** iff for every $x, y \in S, x \bullet y = y \bullet x$.

   - **Axiom 3(a):** B is commutative wrt the operator $+$
   - **Axiom 3(b):** B is commutative wrt the operator $\cdot$

Algebric Properties: Axioms 4–6

4. **Distributivity Property**: If $\bullet$ and $\diamond$ are two binary operators on a set S, $\bullet$ is said to be **distributive** over $\diamond$ if for all $x, y, z \in S, x \bullet (y \diamond z) = (x \bullet y) \diamond (x \bullet z)$.

   - **Axiom 4(a):** The operator $\cdot$ is distributive over $+$; **Axiom 4(b):** The operator $+$ is distributive over $\cdot$.

5. **Complement Element**: For every $x \in B$, there exists an element $x' \in B$ such that $x + x' = 1$ and $x \cdot x' = 0$. x' is called the **complement** of x.

6. **Cardinality Bound**: There are at least two elements $x, y \in B$ such that $x \neq y$.

- Other notes:

  - The **Associative Law** $x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ hold and can be derived from the above axioms.

  - Axioms 4(b) and 5 are not available in ordinary algebra.

  - Boolean algebra applies to a finite set of elements; ordinary algebra applies to the infinite set of real numbers.

Duality and Basic Identities

- **Duality Property:** Every algebraic identity deducible from the previous axioms remains valid after **dual** operations.
  - The **DUAL** of an algebraic expression is formed by interchanging AND $(\cdot) \longleftrightarrow$ OR $(+)$ and $0 \longleftrightarrow 1$, with all previous operation orderings maintained.
  - Exp: $x + y = y + x \longleftrightarrow x \cdot y = y \cdot x$; $x + x' = 1 \longleftrightarrow x \cdot x' = 0$.

    $[f(x_1, x_2, \cdots, x_n, 0, 1, +, \cdot)]^D = f(x_1, x_2, \cdots, x_n, 1, 0, \cdot, +)$
- $\alpha = \beta$ iff $\alpha^D = \beta^D$
- Identities of Boolean algebra:

1.	$x + 0 = x$	2.	$x \cdot 1 = x$	
3.	$x + 1 = 1$	4.	$x \cdot 0 = 0$	
5.	$x + x = x$	6.	$x \cdot x = x$	
7.	$x + \bar{x} = 1$	8.	$x \cdot \bar{x} = 0$	
9.	$\bar{\bar{x}} = x$			
10.	$x + y = y + x$	11.	$xy = yx$	Commutative
12.	$x + (y + z) = (x + y) + z$	13	$x(yz) = (xy)z$	Associative
14.	$x(y + z) = xy + xz$	15.	$x + yz = (x + y)(x + z)$	Distributive
16.	$\overline{x + y} = \bar{x} \cdot \bar{y}$	17.	$\overline{x \cdot y} = \bar{x} + \bar{y}$	DeMorgan's
18.	$x + xy = x$	19.	$x(x + y) = x$	Absorption
20.	$xy + x\bar{y} = x$	21.	$(x + y)(x + \bar{y}) = x$	
22.	$x + \bar{x}y = x + y$	23.	$x(\bar{x} + y) = xy$	

- **Operator precedence:** parenthese () $\rightarrow$ NOT $\rightarrow$ AND $\rightarrow$ OR.

DeMorgan's Law

- Truth tables to verify DeMorgan's Law: $\overline{X+Y} = \bar{X} \cdot \bar{Y}$

A)	X	Y	X+Y	$\overline{X+Y}$	B)	X	Y	$\overline{X}$	$\overline{Y}$	$\overline{X}\bullet\overline{Y}$
	0	0	0	1		0	0	1	1	1
	0	1	1	0		0	1	1	0	0
	1	0	1	0		1	0	0	1	0
	1	1	1	0		1	1	0	0	0

- DeMorgan's Law can be extended to multiple variables.
  - $\overline{X_1 + X_2 + \ldots + X_n} = \bar{X}_1 \bar{X}_2 \ldots \bar{X}_n$.
  - $\overline{X_1 X_2 \ldots X_n} = \bar{X}_1 + \bar{X}_2 + \ldots + \bar{X}_n$.

Algebraic Manipulation to Minimize Literals

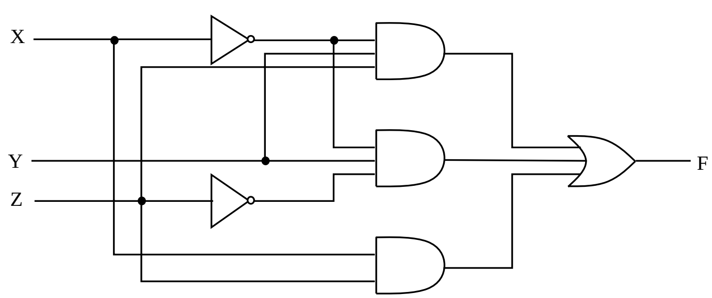
- **Literal:** A single variable within a **term** that may or may not be complemented.

- Minimize # of terms and literals: $\bar{x}yz + \bar{x}y\bar{z} + xz$

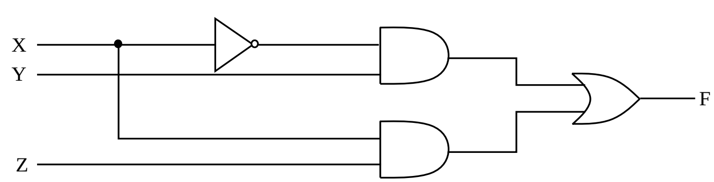
$$
\begin{aligned}
F &= \bar{x}yz + \bar{x}y\bar{z} + xz & \\
&= \bar{x}y(z + \bar{z}) + xz & \text{Identity 14} \\
&= \bar{x}y \cdot 1 + xz & \text{Identity 7} \\
&= \bar{x}y + xz & \text{Identity 2}
\end{aligned}
$$

- # of terms reduced from 3 ($\bar{x}yz, \bar{x}y\bar{z}, xz$) to 2 ($\bar{x}y, xz$).

- # of literals reduced from 8 ($\bar{x}, y, z, \bar{x}, y, \bar{z}, x, z$) to 4 ($\bar{x}, y, x, z$).



(a) $F=\bar{X}YZ+\bar{X}Y\bar{Z}+XZ$



(b) $F=\bar{X}Y+XZ$

Consensus Theorem

(notion: A consensus term is a "redundant" term that can be added or removed without changing functionality.)

- $xy + \bar{x}z + yz = xy + \bar{x}z$

$$\begin{aligned} xy + \bar{x}z + yz &= xy + \bar{x}z + yz(x + \bar{x}) \\ &= xy + \bar{x}z + xyz + \bar{x}yz \\ &= xy + xyz + \bar{x}z + \bar{x}yz \\ &= xy(1 + z) + \bar{x}z(1 + y) \\ &= xy + \bar{x}z \end{aligned}$$

- Dual: $(x + y)(\bar{x} + z)(y + z) = (x + y)(\bar{x} + z)$

- Exp: $(A + B)(\bar{A} + C) = AC + \bar{A}B$

$$\begin{aligned} (A + B)(\bar{A} + C) &= A\bar{A} + AC + \bar{A}B + BC \\ &= AC + \bar{A}B + BC \\ &= AC + \bar{A}B \end{aligned}$$

Function Complement

- Use DeMorgan's Law to complement a function
  1. Interchange AND and OR operators (**duality operation**).
  2. Complement each literal.

- Applying DeMorgan's Law: $F = \bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z$

$$\begin{aligned}\bar{F} &= \overline{\bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z} = \overline{\bar{X}Y\bar{Z}} \cdot \overline{\bar{X}\bar{Y}Z} \\ &= (X + \bar{Y} + Z)(X + Y + \bar{Z})\end{aligned}$$

- Taking duals and complementing literals: $F = \bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z$

$$\begin{aligned}F \quad & = \bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z \\ \text{Taking dual:} \quad & (\bar{X} + Y + \bar{Z})(\bar{X} + \bar{Y} + Z) \\ \text{Complementing literals:} \quad & (X + \bar{Y} + Z)(X + Y + \bar{Z}) = \bar{F}\end{aligned}$$

Function Complement

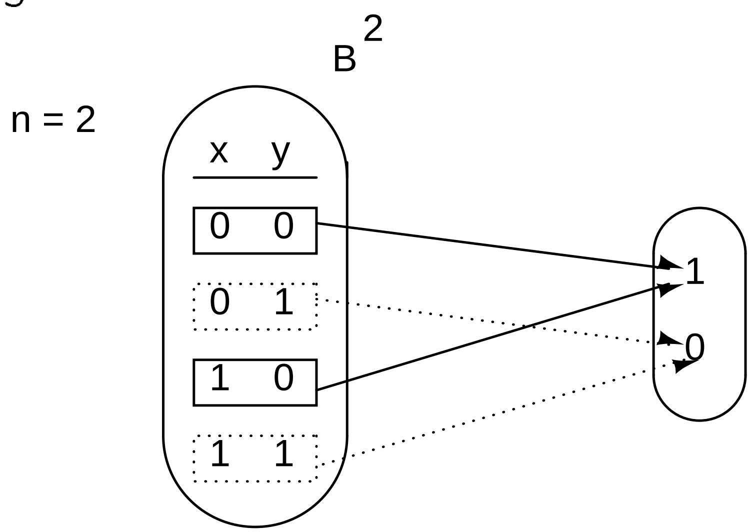
Example:

$$
\begin{aligned}
F &= \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}B\bar{C}\bar{D} + \bar{A}BD + \bar{A}B\bar{C}D + ABCD + AC\bar{D} + \bar{B}C\bar{D} \\
&\text{(Let } x = \bar{A}\bar{C}\bar{D},\ y = \bar{A}BD\text{)} \\
&= x\bar{B} + xB + y + y\bar{C} + ABCD + AC\bar{D} + \bar{B}C\bar{D} \\
&=
\end{aligned}
$$

What is a Boolean function?

$$B^n \xrightarrow{f} B, B = 1, 0, n : \# \text{ of vars}$$

e.g.



- $f(0,0) = f(1,0) = 1$
- $f = \bar{x}\bar{y} + x\bar{y} = m_0 + m_2$ or $f = (x+\bar{y})(\bar{x}+\bar{y}) = M_1 \cdot M_3$
- e.g. If $g(x,y,z) = m_0 + m_1 + m_2 + m_5 + m_7 \Rightarrow g = M_3 \cdot M_4 \cdot M_6$

(maxterm) M_i	(minterm) m_i	x	y	f
$x+y(M_0)$	$\bar{x}\bar{y}(m_0)$	0	0	1
$x+\bar{y}(M_1)$	$\bar{x}y(m_1)$	0	1	0
$\bar{x}+y(M_2)$	$x\bar{y}(m_2)$	1	0	1
$x+y(M_3)$	$xy(m_3)$	1	1	0

What is a Boolean function?

e.g. $n = 3$, 2^3 truth table entries, 2^3 mts, 2^3 Mts.

id	x	y	z	F	$\bar{F}$
0	0	0	0	0	1
1	0	0	1	1	0
2	0	1	0	0	1
3	0	1	1	0	1
4	1	0	0	1	0
5	1	0	1	1	0
6	1	1	0	1	0
7	1	1	1	1	0

- $F = m_1 + m_4 + m_5 + m_6 + m_7 = M_0 \cdot M_2 \cdot M_3$
- $F = \sum_m(1,4,5,6,7) = \prod_M(0,2,3)$
- $\bar{F} = m_0 + m_2 + m_3 = M_1 \cdot M_4 \cdot M_5 \cdot M_6 \cdot M_7$

Minterm and Maxterm

- **Minterms:** Terms with all variables present, combined with AND.
  1. For n variables combined with AND, there are 2^n combinations. Each unique combination is called a **minterm**.
  2. Exp: $X \cdot Y \cdot Z$, $\bar{a}bc\bar{d}$.
- **Maxterms:** Terms with all variables present, combined with OR.
  1. For n variables combined with OR, there are 2^n combinations. Each unique combination is called a **maxterm**.
  2. Exp: $X + Y + Z$, $\bar{a} + b + c + \bar{d}$.
- Exp: Two-variable terms

Index	Minterm	Maxterm
0	$\bar{x}\bar{y}$	$x + y$
1	$\bar{x}y$	$x + \bar{y}$
2	$x\bar{y}$	$\bar{x} + y$
3	xy	$\bar{x} + \bar{y}$

Standard Order

- Given n variables, we use an n-bit expansion of the index, i, to indicate normal (true) or complement states for the variables.

- Minterms: "1" $\Rightarrow$ true; "0" $\Rightarrow$ complemented.

  - m_0 (minterm 0): $\bar{x} \cdot \bar{y} \cdot \bar{z}$; m_3 (minterm 3): $\bar{x} \cdot y \cdot z$.

- Maxterms: "0" $\Rightarrow$ true; "1" $\Rightarrow$ complemented.

  - M_0 (maxterm 0): $x + y + z$; M_3 (maxterm 3): $x + \bar{y} + \bar{z}$.

- m_i is the **complement** of M_i ($m_i = \bar{M}_i$).

Minterms and Maxterms

- Minterms for three variables.

X	Y	Z	Product Term	Symbol	m0	m1	m2	m3	m4	m5	m6	m7
0	0	0	$\overline{X}\,\overline{Y}\,\overline{Z}$	m0	1	0	0	0	0	0	0	0
0	0	1	$\overline{X}\,\overline{Y}Z$	m1	0	1	0	0	0	0	0	0
0	1	0	$\overline{X}Y\overline{Z}$	m2	0	0	1	0	0	0	0	0
0	1	1	$\overline{X}YZ$	m3	0	0	0	1	0	0	0	0
1	0	0	$X\overline{Y}\,\overline{Z}$	m4	0	0	0	0	1	0	0	0
1	0	1	$X\overline{Y}Z$	m5	0	0	0	0	0	1	0	0
1	1	0	$XY\overline{Z}$	m6	0	0	0	0	0	0	1	0
1	1	1	XYZ	m7	0	0	0	0	0	0	0	1

- Maxterms for three variables.

X	Y	Z	Sum Term	Symbol	M0	M1	M2	M3	M4	M5	M6	M7
0	0	0	$X+Y+Z$	M0	0	1	1	1	1	1	1	1
0	0	1	$X+Y+\overline{Z}$	M1	1	0	1	1	1	1	1	1
0	1	0	$X+\overline{Y}+Z$	M2	1	1	0	1	1	1	1	1
0	1	1	$X+\overline{Y}+\overline{Z}$	M3	1	1	1	0	1	1	1	1
1	0	0	$\overline{X}+Y+Z$	M4	1	1	1	1	0	1	1	1
1	0	1	$\overline{X}+Y+\overline{Z}$	M5	1	1	1	1	1	0	1	1
1	1	0	$\overline{X}+\overline{Y}+Z$	M6	1	1	1	1	1	1	0	1
1	1	1	$\overline{X}+\overline{Y}+\overline{Z}$	M7	1	1	1	1	1	1	1	0

Sum of Minterms

- Any Boolean function can be expressed as a sum of minterms.
- Exp: $F = A + \bar{B}C$.
  - Expand the terms with missing variables and collect terms:

$$\begin{aligned} F &= A + \bar{B}C \\ &= A(B+\bar{B})(C+\bar{C}) + (A+\bar{A})\bar{B}C \\ &= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + \bar{A}\bar{B}C \end{aligned}$$

  - Sum of minterms: $F = m_7 + m_6 + m_5 + m_4 + m_1 \Longrightarrow F(A,B,C) = \sum_m(1,4,5,6,7)$.
- The complement of a function contains those minterms not included in the original function.
  - $F(A,B,C) = \sum_m(1,4,5,6,7) \Longrightarrow \bar{F}(A,B,C) = \sum_m(0,2,3)$.

Product of Maxterms

- Any Boolean function can be expressed as a product of maxterms.
- Exp: $G = AB + \bar{A}\bar{B}$.

$$\begin{aligned} AB + \bar{A}\bar{B} &= (AB + \bar{A})(AB + \bar{B}) \\ &= (\bar{A} + AB)(\bar{B} + AB) \\ &= (\bar{A} + A)(\bar{A} + B)(\bar{B} + A)(\bar{B} + B) \\ &= 1 \cdot (\bar{A} + B)(\bar{B} + A) \cdot 1 \\ &= (\bar{A} + B)(\bar{B} + A) \\ &= M_2 \cdot M_1 \end{aligned}$$

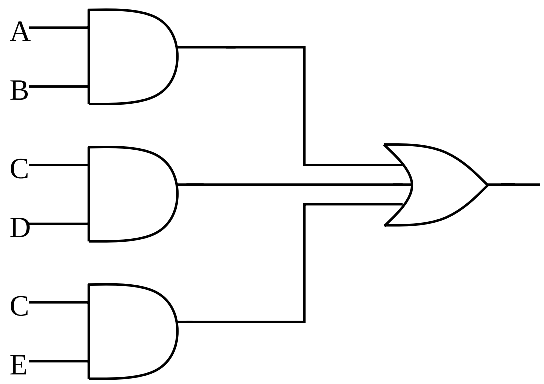
- Product of maxterms: $G = M_2 \cdot M_1 \Longrightarrow G(A,B) = \Pi_M(1,2)$.
- The complement of a function contains those maxterms not included in the original function.
  - $G(A,B) = \Pi_M(1,2) \Longrightarrow \bar{G}(A,B) = \Pi_M(0,3)$.
- $G(A,B) = \Pi_M(1,2) = \sum_m(0,3)$; $\bar{G}(A,B) = \Pi_M(0,3) = \sum_m(1,2)$.

Standard Forms

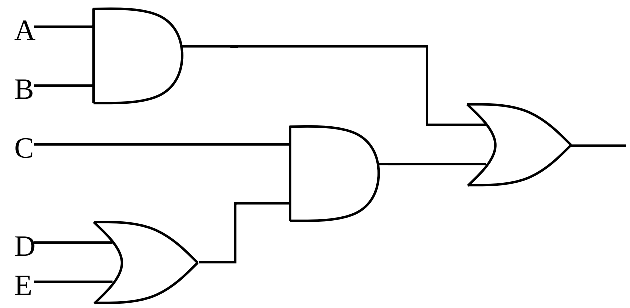
- **Canonical forms** (Sum-of-Minterms or Product-of-Maxterms) have **one and only one** representation.
  - Sum-of-Minterms: $xyz + \bar{x}y\bar{z} + x\bar{y}z$, $A\bar{B} + \bar{A}B$.
  - Product-of-Maxterms: $(x+y+z)(\bar{x}+y+\bar{z})(x+\bar{y}+z)$, $(A+\bar{B})(\bar{A}+B)$
- **Standard Sum-of-Products (SOP) form:** Equations are written as **AND** terms summed with **OR** operators.
  - SOPs: $xyz + \bar{x}y\bar{z} + \bar{y}$, $A\bar{B} + \bar{A}B$
- **Standard Product-of-Sums (POS) form:** Equations are written as **OR** terms, all **AND**ed together.
  - POSs: $(x + y + z)(\bar{x} + y + \bar{z})(\bar{y})$, $(A + \bar{B})(\bar{A} + B)$
- Mixed forms: **not** SOP or POS
  - $(x\bar{y} + z)(\bar{x} + y)$, $AB\bar{C} + C(\bar{A} + B)$

Canonical Sum of Minterms

- Two-level vs. three-level implementation.
  - Exp: $F = AB + CD + CE$ (2-level); $F = AB + C(D + E)$ (3-level)



(a) AB+CD+CE

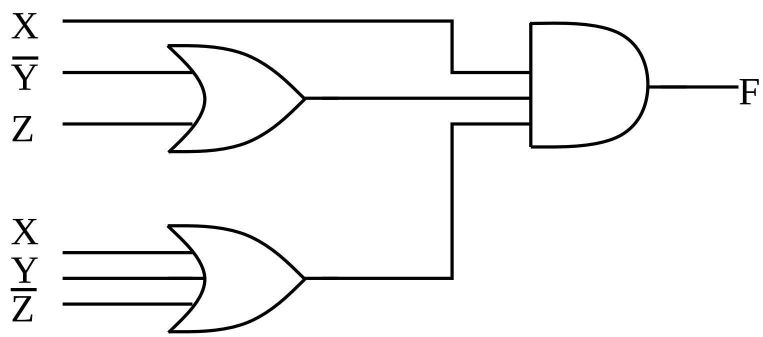


(b) AB+C(D+E)

- A canonical sum-of-minterms form implies a 2-level network of gates implementation (**1st level: AND; 2nd level: OR**).
  - Usually **not** a minimum literal Boolean expression $\Longrightarrow$ more expensive implementation.
  - Exp: $F = ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + \bar{A}\bar{B}C = A + \bar{B}C$ (Why? Duplicate $A\bar{B}C$!)
  - The canonical sum-of-minterms form had **15 literals** and **5 terms**; the reduced SOP form had **3 literals** and **2 terms**.

Canonical Product of Maxterms

- Two-level product-of-sums implementation.
  - Exp: $F = X(\bar{Y} + Z)(X + Y + \bar{Z})$



- A canonical product-of-maxterms form implies a 2-level network of gates implementation (**1st level: OR; 2nd level: AND**).
  - Usually **not** a minimum literal Boolean expression $\Longrightarrow$ more expensive implementation.
  - Exp: $F = (A + B + C)(A + \bar{B} + C)(A + \bar{B} + \bar{C}) = (A + C)(A + \bar{B})$ (Why? Duplicate $(A + \bar{B} + C)$!)
  - The canonical product-of-maxterms form had **9 literals** and **3 terms**; the reduced POS form had **4 literals** and **2 terms**.

Equivalent Literal Cost Network

- **Questions:**
  - Is there only one minimum cost network?
  - How can we obtain **a** or **the** minimum literal expression?
- Minimize $F(A,B,C) = \sum(0,2,3,4,5,7)$

$$
\begin{aligned}
F &= \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + A\bar{B}\bar{C} + A\bar{B}C + ABC \\
&= \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + ABC + A\bar{B}\bar{C} + A\bar{B}C \\
&= \bar{A}\bar{C}(B + \bar{B}) + BC(\bar{A} + A) + A\bar{B}(C + \bar{C}) \\
&= \bar{A}\bar{C} + BC + A\bar{B}
\end{aligned}
$$

- Pairing F's terms differently

$$
\begin{aligned}
F &= \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + A\bar{B}\bar{C} + A\bar{B}C + ABC \\
&= \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + \bar{A}BC + \bar{A}B\bar{C} + A\bar{B}C + ABC \\
&= \bar{B}\bar{C}(\bar{A} + A) + \bar{A}B(C + \bar{C}) + AC(B + \bar{B}) \\
&= \bar{B}\bar{C} + \bar{A}B + AC
\end{aligned}
$$

- **Both have the same numbers of literals and terms!**