An Alternative To The Iteration Operator Of Propositional Dynamic Logic

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Abstract

In this paper we present an alternative to the iteration operator of Propositional Dynamic Logic (PDL) and a tableau system for the resulting logic. We show the correctness and completeness of the method.

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1 Introduction

In this work we present an alternative to the iteration operator $(*)$ of Propositional Dynamic Logic (PDL) [Har84]. The way we do this is to replace this operator by the necessity operator $(\Box)$ of modal logic $S4$ plus an interaction axiom of multimodal logic. The resulting logic (called $\mathcal{A}$) is weaker than PDL, but is strong enough to reason about actions and plans.

In section 2 we present the language, semantics and axiomatics of $\mathcal{A}$. In section 3 we show how one can reason about actions and plans in $\mathcal{A}$. In section 4 we present a tableau system for $\mathcal{A}$, to provide an automatic way of deciding whether a formula is a theorem or not. Sections 5 and 6 present, respectively, the correctness and completeness results for the tableau system.

2 The logic $\mathcal{A}$

In this section we present the logic $\mathcal{A}$, the language, semantics and axiomatics. We also present the notation to be used in the rest of the paper.

2.1 Language

The language of $\mathcal{A}$ is as follows: Let ACT_0 be a set of *atomic actions* ($ACT_0 = \{\alpha, \beta, \ldots\}$) and let FOR_0 be a set of *atomic formulas*, or just *atoms* ($FOR_0 = \{P, Q, \ldots\}$). We construct inductively the set ACT of *actions* and the set FOR of *formulas* in the following way:

- $\top \in FOR$; $\bot \in FOR$; $FOR_0 \subseteq FOR$
- if $A, B \in FOR$ then $\neg A \in FOR$, $A \vee B \in FOR$ and $\Box A \in FOR$.
- if $A \in FOR$ and $\alpha \in ACT$ then $[\alpha]A \in FOR$
- $ACT_0 \subseteq ACT$;
- $\lambda \in ACT_0$
- if $\alpha, \beta \in ACT$ then $(\alpha;\beta) \in ACT$ and $(\alpha \cup \beta) \in ACT$
- if $A \in FOR$ then $A? \in ACT$

We use $\wedge$, $\rightarrow$, $\leftrightarrow$ as usual, $\langle\alpha\rangle A$ to abbreviate $\neg[\alpha]\neg A$ and $\Diamond A$ to abbreviate $\neg\Box\neg A$ as in modal logic. We also use $[\alpha;\beta]A$ for $[\alpha][\beta]A$, $[\alpha \cup \beta]A$ for $([\alpha]A \wedge [\beta]A)$, $[A?]B$ for $(A \rightarrow B)$, $[\lambda]A$ for A and $[\text{if } A \text{ then } \alpha \text{ else } \beta]B$ for $((A \rightarrow [\alpha]B) \wedge (\neg A \rightarrow [\beta]B))$

As in dynamic logic we read λ "do nothing", $A?$ "continue if A holds", $\alpha \cup \beta$ "do either α or β nondeterministically", $(\alpha;\beta)$ "do α followed by β" and $[\alpha]A$ "every possible execution of action α leads to a situation in which A is true". We read $\Box A$ "all *plans* leads to a situation in which A is true". A plan is a (possibly empty) sequence of actions. Finally, if a formula does not contain modal operators, we call it a classical formula.

2.2 Semantics

We define *standard models* for our logic $\mathcal{A}$ as quadruples of the form $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box, \models)$, where W is a set of worlds, each R_α and $R_\Box \subseteq W \times W$ is a binary relation (called an accessibility relation) and $\models$ is the satisfaction relation with the following truth condition:

- $\models_w^\mu P$ iff $w \in \tau(P)$, where $\tau : FOR_0 \longrightarrow 2^W$
- $\models_w^\mu \top$
- not $\models_w^\mu \bot$
- $\models_w^\mu \neg A$ iff *not* $\models_w^\mu A$
- $\models_w^\mu A \wedge B$ iff $\models_w^\mu A$ and $\models_w^\mu B$
- $\models_w^\mu A \rightarrow B$ iff if $\models_w^\mu A$ then $\models_w^\mu B$
- $\models_w^\mu A \vee B$ iff $\models_w^\mu A$ or $\models_w^\mu B$
- $\models_w^\mu A \leftrightarrow B$ iff $\models_w^\mu A$ if and only if $\models_w^\mu B$
- $\models_w^\mu [\alpha]A$ iff $\forall w' \in W$ such that $wR_\alpha w'$, $\models_{w'}^\mu A$.
- $\models_w^\mu \Box A$ iff $\forall w' \in W$ such that $wR_\Box w'$, $\models_{w'}^\mu A$.

We impose no restrictions on the (R_α) family of relations, but we require $R_\Box$ to be reflexive and transitive. Moreover, we suppose $R_\Box \supseteq R_\alpha, \alpha \in ACT_0$.

Validity and satisfiability are defined as usual. We denote $\models_{\mathcal{A}} P$ to mean "P is valid in $\mathcal{A}$".

2.3 Axiomatics

The following are the axioms and rules of inference schemes for $\mathcal{A}$, where $A, B \in FOR$ and $\alpha \in ACT$:

PL**:** All tautologies of classical propositional calculus

$K(\alpha)$**:** $[\alpha](A \rightarrow B) \rightarrow ([\alpha]A \rightarrow [\alpha]B)$

$K(\Box)$**:** $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

$T(\Box)$**:** $\Box A \rightarrow A$

$4(\Box)$**:** $\Box A \rightarrow \Box\Box A$

$I(\Box, \alpha)$**:** $\Box A \rightarrow [\alpha]A$

$(M.P.)$**:** if A and $A \rightarrow B$ then infer B

($RN_\Box$**):** if A then infer $\Box A$

We denote $\vdash_{\mathcal{A}} A$ to mean "A is a theorem of $\mathcal{A}$" and theoremhood is defined as usual.

Remark.: $\mathcal{A}$ may be seen as a multimodal logic made up from Propositional Dynamic Logic (PDL) [Har84] without the iteration operator (*), plus the well known modal logic $S4$ for the $\Box$-operator plus the interaction axioms $\{I(\Box, \alpha) : \Box A \rightarrow [\alpha]A\}$. In fact, $\mathcal{A}$ is an abbreviation for $K(\alpha) + S4(\Box) + I(\Box, \alpha)$.

3 Reasoning about actions and plans in $\mathcal{A}$

In this section we show how the components of systems for reasoning about actions and plans may be written in the formalism of our logic $\mathcal{A}$. We present usual tasks and classical problems of reasoning about actions and plans in terms of $\mathcal{A}$, but we don't propose any solution for them. This is subject of future research and is presented now only as a motivation.

3.1 Representation

Let ACT_0 stand for *atomic actions* and ACT stand for *compound actions* built with elements of ACT_0 and the operators "λ", "?", ";", "$\cup$", "if ... then ... else ...". For example, we may have atomic actions like "*load*", "*shoot*" and "*wait*", and more complicated compound actions like "$load; wait; shoot$" and "if $dead$ then λ else $load; shoot$".

Let KB be the *knowledge base*, the set of classical formulas representing the actual world. Let IC_{ACT} be the set of *Action Laws*. They have the form $\Box(A \rightarrow [\alpha]B)$, where A is the *precondition* and B is the *postcondition* of the action α. Special action laws have the form $\Box(A \rightarrow [\alpha]A)$ and are called *frame axioms*. Finally, we have the set IC_{CLASS} of *Integrity Constraints*, whose elements have the form $\Box A$, where A is classical. In planning systems we also have the set *Goal*, whose formulas have the form $\Diamond A$, where A is classical.

Example:

KB :
$walking \wedge loaded.$

IC_{CLASS} :
$\Box(alive \leftrightarrow \neg dead),$
$\Box(loaded \leftrightarrow \neg empty),$
$\Box(walking \rightarrow alive).$

IC_{ACT} :
$\langle wait \rangle \top$
$\langle load \rangle \top$
$loaded \rightarrow [shoot]dead,$
$loaded \rightarrow [shoot]empty,$
(frame axioms:)
$alive \rightarrow [wait]alive,$
$loaded \rightarrow [wait]loaded$
$walking \rightarrow [load]walking,$
$dead \rightarrow [shoot]dead$

$Goal$: $\Diamond dead$, $\Diamond alive$

3.2 Reasoning tasks

Once we have a representation for the world and for the actions laws, the essentials tasks we are interested includes:

1. *Planning*: We know the actual and the desired world, and we want to know about the existence of a sequence of actions that lead from the actual situation to the desired one. In other words we want to know whether $\vdash_{\mathcal{A}} KB \wedge IC_{ACT} \wedge IC_{CLASS} \rightarrow Goal$ is a theorem of $\mathcal{A}$. We claim that as antecedents do not contain positive $\Diamond-$operators, from the proof of a planning problem we can construct a plan. For example, we may have to know if there is a proof of the formula "$\neg loaded \wedge \Box(loaded \rightarrow [shoot]dead) \wedge \Box[load]loaded \rightarrow \Diamond dead$". The proof of this theorem gives us a plan.

2. *Prediction*: We know the actual world (KB_1) and we want to point out the correct state of the world we may obtain (KB_2) if we apply in KB_1 a given sequence of actions. In the formalism we have, what we want to know is whether $\vdash_{\mathcal{A}} KB_1 \wedge IC_{ACT} \wedge IC_{CLASS} \rightarrow [\alpha]KB_2$ is a theorem of $\mathcal{A}$.

3. *Postdiction*: We want to know the preceding state of the world (KB_1) having only the actual world (KB_2) and knowing that a given sequence of actions was executed. In other words, we want to know whether $\vdash_{\mathcal{A}} [\alpha]KB_2 \wedge IC_{ACT} \wedge IC_{CLASS} \rightarrow KB_1$ is a theorem of the logic.

3.3 Reasoning problems

In order to realize tasks as those described in the previous section, three main problems arise. These problems were pointed out the first time by McCarthy and Hayes [MH69]:

1. The *frame problem* is that in real world, it is impractical to write down all non-consequences of an action. In other words, after an action, one must say everything that do not change in the world. In terms of $\mathcal{A}$, the frame problem is that the set of frame axioms is too big, and in fact it contain almost all formulas of IC_{ACT}. E.g. we don't want to write the frame axioms $alive \rightarrow [wait]alive$, $loaded \rightarrow [wait]loaded$, etc...

2. The *ramification problem* is the dual of the frame problem, i.e., it is difficult to take into account all the (direct and indirect) consequences of an action, i.e., it is difficult to take into account the set IC_{CLASS}. E.g. after the execution of $[shoot]$ in $loaded \rightarrow [shoot]dead$, one must also conclude that $\neg walking$ and $\neg loaded$.

3. The *qualification problem* is related with the difficulty to write down the potentially immense number of preconditions of an action. E.g. the preconditions of the action of starting a car. Moreover, it may be computationally intractable to check all qualifications for each action to be executed.

We note that with our formalism it is possible to solve the ramification problem. In fact the solution is straightforward since we have the set IC_{CLASS}. We don't have solutions to the frame neither the qualification problem, but for the former we claim that it is possible to find such a solution following the work of Gasquet and Herzig [GH95].

4 $\mathcal{A}$-Tableau rules

In this section we present a tableau system for $\mathcal{A}$. We will use a generalization of Fitting's signed formulas notation [Fit83].

Thus, if A is a formula we write:

- $\mathcal{V}^{\Box}$ to mean $T\Box A$ or $F\Diamond A$
- $\mathcal{V}^{\alpha}$ to mean $T[\alpha]A$ or $F\langle\alpha\rangle A$
- $\pi^{\Box}$ to mean $T\Diamond A$ or $F\Box A$
- π^{α} to mean $T\langle\alpha\rangle A$ or $F[\alpha]A$

Moreover, the result of removing the modal operator of $\mathcal{V}^{\Box}$, $\mathcal{V}^{\alpha}$, $\pi^{\Box}$or π^{α} is denoted by a subscripted 0. For example, if $\mathcal{V}^{\Box} = \Box A$ then $\mathcal{V}^{\Box}_0 = A$.

We also consider the following definitions in order to simplify notation in the next section. Let S be a set of formulas and $\alpha \in ACT$. Then

- $S^{\Box} = \{\mathcal{V}^{\Box} : \mathcal{V}^{\Box} \in S\}$
- $S^{\alpha} = \{\mathcal{V}^{\alpha}_0 : \mathcal{V}^{\alpha} \in S\}$

When we want to evidence some formula A in a set of formulas S we will write $S, \{A\}$. In this case, we consider that $A \notin S$.

The $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-rules[2] are the same as in [Fit83] and are not listed. The other rules are divided in two categories: those that do not correspond to a world change in the semantics (the $\mathcal{V}$-rules) and those that do correspond to a world change in the semantics (the π-rules, represented in the figures by an arrow). In the former, like the $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-rules, we simply add a formula to the set, and in the latter we erase all formulas and rewrite new ones in the new world. They are[3]:

[2] $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ appear in boldface when they are used as rule names and not to denote dynamic operators.
[3] This method does not have a $\mathcal{V}^{\alpha}$-rule.

$\mathcal{V}^{\Box}$-rule	$\pi^{\Box}$-rule	π^{α}-rule
from $\mathcal{V}^{\Box}$	from $S, \pi^{\Box}$	from S, π^{α}
	$\downarrow$	$\downarrow$
infer $\mathcal{V}_0^{\Box}$	infer $S^{\Box}, \pi_0^{\Box}$	infer $S^{\Box}, S^{\alpha}, \pi_0^{\alpha}$

Example: If we have the following formulas in a world w, where $\alpha, \beta \in ACT$, $A_i \in FOR$:

w

1. $T\Diamond A_1$
2. $T\Diamond A_2$
3. $T\Box A_3$
4. $T\Box A_4$
5. $T[\alpha]A_5$
6. $T[\beta]A_6$
7. $T\langle\alpha\rangle A_7$
8. $T\langle\beta\rangle A_8$

We can apply the $\mathcal{A}$-tableau rules to obtain (in the same world):

w

9.	TA_3	3, $\mathcal{V}^{\Box}$-rule
10.	TA_4	4, $\mathcal{V}^{\Box}$-rule

or we can apply the $\mathcal{A}$-tableau rules to obtain (in different worlds w_1 and w_2):

w_1			w_2	
1'. $T\Box A_3$			1". $T\Box A_3$	
2'. $T\Box A_4$	1, $\pi^{\Box}$-rule	or	2". $T\Box A_4$	7, π^{α}-rule
3'. TA_1			3". TA_5	
			4". TA_7	

5 Correctness

In this section we will present the correctness proof of the $\mathcal{A}$-tableau system. In order to do this we have to show two lemmas.

Definition 1 If w is a world in a model $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_{\Box}, \models)$ and S is a set of signed formulas, we write $\models_w^\mu S$ when $\models_w^\mu A$, $\forall A \in S$.

Lemma 1 Suppose $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_{\Box}, \models)$ is a standard model, S is a set of signed formulas, $w \in W$ and $w' \in W$ is any world such that $wR_\alpha w'$ (and hence $wR_{\Box}w'$, since $R_{\Box} \supseteq R_\alpha$). Then:

1. $\models_w^\mu S$ implies $\models_{w'}^\mu S^\alpha \cup \{\mathcal{V}_0^{\Box} : \mathcal{V}^{\Box} \in S\}$.
2. $\models_w^\mu S$ implies $\models_{w'}^\mu S^{\Box}$.
3. $\models_w^\mu \mathcal{V}^{\Box}$ implies $\models_w^\mu \mathcal{V}_0^{\Box}$.

Proof 1. If $\models_w^\mu S$ then $\models_w^\mu A$, $\forall A \in S$. If A is of the form $\mathcal{V}^\alpha$, then by definition, $\forall w' \in \mu$, $wR_\alpha w'$, $\models_{w'}^\mu \mathcal{V}_0^\alpha$. So $\models_{w'}^\mu \{\mathcal{V}_0^\alpha | \mathcal{V}^\alpha \in S\}$, i.e., $\models_{w'}^\mu S^\alpha$. And since $R_{\Box} \supseteq R_\alpha, \forall \alpha \in ACT$ we have $wR_{\Box}w'$. So for each $\mathcal{V}^{\Box} \in S$ we have by definition $\models_{w'}^\mu \mathcal{V}_0^{\Box}$. Thus $\models_{w'}^\mu \{\mathcal{V}_0^{\Box} : \mathcal{V}^{\Box} \in S\}$. Hence $\models_{w'}^\mu S^\alpha \cup \{\mathcal{V}_0^{\Box} : \mathcal{V}^{\Box} \in S\}$.

2. Let w'' be any world in W such that $w'R_\Box w''$. Since $wR_\Box w'$ and $R_\Box$ is transitive, $wR_\Box w''$. So, for each $\mathcal{V}^\Box \in S$, if $\models^\mu_w \mathcal{V}^\Box$ it follows $\models^\mu_{w'} \mathcal{V}^\Box_0$, i.e., $\forall w'' \in \mu$ such that $w'R_\Box w''$ we have $wR_\Box w''$ and so $\models^\mu_{w''} \mathcal{V}^\Box_0$, in other words, $\models^\mu_{w'} \mathcal{V}^\Box$. Hence $\models^\mu_{w'} S^\Box$.

3. Since $\models^\mu_w \mathcal{V}^\Box$, we have by definition $\models^\mu_{w'} \mathcal{V}^\Box_0$, $\forall w' \in \mu$ such that $wR_\Box w'$. But $R_\Box$ is reflexive, so in particular we have $wR_\Box w$ and hence $\models^\mu_w \mathcal{V}^\Box_0$.

Definition 2 A tableau is $\mathcal{A}$-satisfiable if some branch of it is $\mathcal{A}$-satisfiable. A branch of a tableau is $\mathcal{A}$-satisfiable if the set of formulas on it is $\mathcal{A}$-satisfiable. A set S of formulas is $\mathcal{A}$-satisfiable if there is a world w in a standard model μ such that $\models^\mu_w S$.

Lemma 2 Suppose T is a $\mathcal{A}$-satisfiable tableau. Then T', the tableau that results from a single $\mathcal{A}$-tableau rule applied to T, is also $\mathcal{A}$-satisfiable.

Proof If T is a $\mathcal{A}$-satisfiable tableau then some branch of it is $\mathcal{A}$-satisfiable. If a branch other then one on which we apply the rule is $\mathcal{A}$-satisfiable that branch is still present in T' and is still $\mathcal{A}$-satisfiable. Hence T' is $\mathcal{A}$-satisfiable. Suppose θ is a $\mathcal{A}$-satisfiable branch of T and the tableau rule is applied on θ. We analyze the consequences for each tableau rule:

i) case $\boldsymbol{\alpha}$-rule is applied ($\theta = S, \boldsymbol{\alpha}$ is replaced by $\theta_1 = S, \boldsymbol{\alpha}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$). Now θ is $\mathcal{A}$-satisfiable, so in some $\mathcal{A}$-model $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box, \models)$ there is a world w such that $\models^\mu_w S$ and $\models^\mu_w \alpha$. By definition $\models^\mu_w \boldsymbol{\alpha}_1$ and $\models^\mu_w \boldsymbol{\alpha}_2$, and hence θ_1 is $\mathcal{A}$-satisfiable.

ii) case $\boldsymbol{\beta}$-rule is applied ($\theta = S, \boldsymbol{\beta}$ is replaced by $\theta_1 = S, \boldsymbol{\beta}, \boldsymbol{\beta}_1$ and $\theta_2 = S, \boldsymbol{\beta}, \boldsymbol{\beta}_2$). Now θ is $\mathcal{A}$-satisfiable, so in some $\mathcal{A}$-model $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box, \models)$ there is a world w such that $\models^\mu_w S$ and $\models^\mu_w \boldsymbol{\beta}$. Hence $\models^\mu_w \boldsymbol{\beta}_1$ or $\models^\mu_w \boldsymbol{\beta}_2$, and hence either θ_1 or θ_2 is $\mathcal{A}$-satisfiable.

iii) case $\mathcal{V}^\Box$-rule is applied ($\theta = S, \mathcal{V}^\Box$ is replaced by $\theta_1 = S, \mathcal{V}^\Box, \mathcal{V}^\Box_0$). Now θ is $\mathcal{A}$-satisfiable, so in some $\mathcal{A}$-model $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box, \models)$ there is a world w such that $\models^\mu_w S$ and $\models^\mu_w \mathcal{V}^\Box$. By lemma 1 we also have $\models^\mu_w \mathcal{V}^\Box_0$. Hence θ_1 is $\mathcal{A}$-satisfiable.

iv) case $\pi^\Box$-rule is applied ($\theta = S, \pi^\Box$ is replaced by $\theta_1 = S^\Box, \pi^\Box_0$). Now θ is $\mathcal{A}$-satisfiable, so in some $\mathcal{A}$-model $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box, \models)$ there is a world w such that $\models^\mu_w S$ and $\models^\mu_w \pi^\Box$. But if $\models^\mu_w \pi^\Box$ then, by definition, for some world $w_1 \in \mu$ such that $wR_\Box w_1$ we have $\models^\mu_{w_1} \pi^\Box_0$. Moreover, by lemma 1, if $\models^\mu_w S$ then $\models^\mu_{w_1} S^\Box$. Hence θ_1 is $\mathcal{A}$-satisfiable in $w_1 \in \mu$, so θ_1 is $\mathcal{A}$-satisfiable.

v) case π^α-rule is applied ($\theta = S, \pi^\alpha$ is replaced by $\theta_1 = S^\Box, S^\alpha, \pi^\alpha_0$). Now θ is $\mathcal{A}$-satisfiable, so in some $\mathcal{A}$-model $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box, \models)$ there is a world w such that $\models^\mu_w S$ and $\models^\mu_w \pi^\alpha$. But if $\models^\mu_w \pi^\alpha$ then for some world $w_1 \in \mu$ such that $wR_\alpha w_1$ we have $\models^\mu_{w_1} \pi^\alpha_0$ and by lemma 1, if $\models^\mu_w S$ then $\models^\mu_{w_1} S^\Box$ and $\models^\mu_{w_1} S^\alpha$ (recall $R_\Box \supseteq R_\alpha, \forall \alpha \in ACT$, so if $wR_\alpha w_1$ then $wR_\Box w_1$). So again θ_1 is $\mathcal{A}$-satisfiable in $w_1 \in \mu$. Hence θ_1 is $\mathcal{A}$-satisfiable.

There are no more cases. Hence this completes the proof.

Theorem 1 (Correctness theorem for $\mathcal{A}$) If A has an $\mathcal{A}$-tableau proof then $\models_\mathcal{A} A$.

Proof Suppose there is a tableau proof starting by FA that arrives at a closed tableau. If $\{FA\}$ were $\mathcal{A}$-satisfiable, we would start with an $\mathcal{A}$-satisfiable tableau, and by lemma 2 every $\mathcal{A}$-tableau we get will also be $\mathcal{A}$-satisfiable. But this is a contradiction because a closed tableau is not $\mathcal{A}$-satisfiable. Hence FA is not $\mathcal{A}$-satisfiable, i.e., there is no $\mathcal{A}$-model μ such that $\not\models^\mu_w A$,for some world $w \in \mu$. In other words, for every $\mathcal{A}$-model μ we have $\models^\mu_w A$, that is, $\models_\mathcal{A} A$.

6 Completeness

In this section we will prove the completeness of our tableau system. We will use the method proposed in [Fit83], i.e., we will first show consistency properties for $\mathcal{A}$ (section 6.1), then we will show the model existency theorem for $\mathcal{A}$ (section 6.2), and finally (in section 6.3) we will present the completeness proof.

6.1 Consistency properties for $\mathcal{A}$

In this section we present the definition of $\mathcal{A}$-consistency property and we show two important lemmas.

Definition 3 [$\mathcal{A}$-Consistency Property] Let $\mathcal{C}$ be a collection of sets of signed formulas. $\mathcal{C}$ is an $\mathcal{A}$-consistency property if it meets the following conditions, for each $S \in \mathcal{C}$:

0. S contains no atomic signed formula TA and its conjugate FA; S contains neither $F\top$ nor $T\bot$.
1. $\boldsymbol{\alpha} \in S \Rightarrow S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \in \mathcal{C}$
2. $\boldsymbol{\beta} \in S \Rightarrow S \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}$ or $S \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}$
3. $\pi^\Box \in S \Rightarrow S^\Box \cup \{\pi_0^\Box\} \in \mathcal{C}$
4. $\pi^\alpha \in S \Rightarrow S^\Box \cup S^\alpha \cup \{\pi_0^\alpha\} \in \mathcal{C}$
5. $\mathcal{V}^\Box \in S \Rightarrow S \cup \{\mathcal{V}_0^\Box\} \in \mathcal{C}$

If $S \in \mathcal{C}$ we say S is $\mathcal{C}$-consistent else S is $\mathcal{C}$-inconsistent.

Definition 4 Let $\mathcal{C}$ be an $\mathcal{A}$-consistency property and let U be a set of signed formulas. We call $\mathcal{C}$ U-compatible if, for each $S \in \mathcal{C}$ and for each $Z \in U$, $S \cup \{Z\} \in \mathcal{C}$.

Lemma 3 Let $\mathcal{C}$ be an $\mathcal{A}$-consistency property. Let $\mathcal{C}'$ contain all subsets of all members of $\mathcal{C}$. Then $\mathcal{C}'$ is also an $\mathcal{A}$-consistency property, extending $\mathcal{C}$ and closed under subsets. Also, if $\mathcal{C}$ is U-compatible, so is $\mathcal{C}'$.

Proof We check each item of the definition:

0. Members of $\mathcal{C}'$ are subsets of members of $\mathcal{C}$, which do not contain S with an atomic formula and its conjugate. So $\mathcal{C}'$ either. For the same reason $\mathcal{C}'$ does not contain S with $F\top$ or $T\bot$.
1. Suppose $S \in \mathcal{C}'$ and $\boldsymbol{\alpha} \in S$. Then $\exists T \in \mathcal{C}$, $S \subseteq T$ and, since $\mathcal{C}$ is an $\mathcal{A}$-consistency property, $T \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \in \mathcal{C}$. But, by construction of $\mathcal{C}'$, and the fact $S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \subseteq T \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$, we have $S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \in \mathcal{C}'$.
2. Suppose $S \in \mathcal{C}'$ and $\boldsymbol{\beta} \in S$. Then $\exists T \in \mathcal{C}$, $S \subseteq T$ and, since $\mathcal{C}$ is an $\mathcal{A}$-consistency property, $T \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}$ or $T \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}$. Again by construction of $\mathcal{C}'$, and the fact $S \cup \{\boldsymbol{\beta}_1\} \subseteq T \cup \{\boldsymbol{\beta}_1\}$ and $S \cup \{\boldsymbol{\beta}_2\} \subseteq T \cup \{\boldsymbol{\beta}_2\}$, we have $S \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}'$ or $S \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}'$.
3. Suppose $S \in \mathcal{C}'$ and $\pi^\Box \in S$. Then $\exists T \in \mathcal{C}$, $S \subseteq T$, and then $S^\Box \subseteq T^\Box$ (recall definition of $S^\Box$ in section 4). Now, since $\mathcal{C}$ is an $\mathcal{A}$-consistency property, $T^\Box \cup \{\pi_0^\Box\} \in \mathcal{C}$. Again by construction of $\mathcal{C}'$ and the fact that $S^\Box \cup \{\pi_0^\Box\} \subseteq T^\Box \cup \{\pi_0^\Box\}$, we have $S^\Box \cup \{\pi_0^\Box\} \subseteq \mathcal{C}'$.
4. Suppose $S \in \mathcal{C}'$ and $\pi^\alpha \in S$. Then $\exists T \in \mathcal{C}$, $S \subseteq T$ and since $\mathcal{C}$ is an $\mathcal{A}$-consistency property, $T^\Box \cup T^\alpha \cup \{\pi_0^\Box\} \in \mathcal{C}$. But if $S \subseteq T$ then by the definitions of section 4, $S^\alpha \subseteq T^\alpha$ and $S^\Box \subseteq T^\Box$. Since $S^\Box \cup S^\alpha \cup \{\pi_0^\alpha\} \subseteq T^\Box \cup T^\alpha \cup \{\pi_0^\alpha\}$ and by the definition of $\mathcal{C}'$, we conclude $S^\Box \cup S^\alpha \cup \{\pi_0^\alpha\} \in \mathcal{C}'$.
5. Suppose $S \in \mathcal{C}'$ and $\mathcal{V}^\Box \in S$. Then $\exists T \in \mathcal{C}$, $S \subseteq T$ and since $\mathcal{C}$ is an $\mathcal{A}$-consistency property, $T \cup \{\mathcal{V}_0^\Box\} \in \mathcal{C}$. Since $S \subseteq T$ implies $S \cup \{\mathcal{V}_0^\Box\} \subseteq T \cup \{\mathcal{V}_0^\Box\}$ and by the definition of $\mathcal{C}'$, we conclude $S \cup \{\mathcal{V}_0^\Box\} \in \mathcal{C}'$.

Hence $\mathcal{C}'$ is a $\mathcal{A}$-consistency property. Obviously, $\mathcal{C}'$ extends $\mathcal{C}$ and is closed under subsentences. We show the U-compatibility: Let $S \in \mathcal{C}'$ and $Z \in U$. Now $S \subseteq T \in \mathcal{C}$ and hence $T \cup \{Z\} \in \mathcal{C}$ (which is U-compatible). It follows that $S \cup \{Z\} \subseteq T \cup \{Z\} \in \mathcal{C}'$. This concludes the proof that $\mathcal{C}'$ is U-compatible.

Lemma 4 Let $\mathcal{C}'$ be an $\mathcal{A}$-consistency property closed under subsets. Let $\mathcal{C}''$ consist of those sets S all of whose finite subsets are in $\mathcal{C}'$. Then $\mathcal{C}''$ is also an $\mathcal{A}$-consistency property, extending $\mathcal{C}'$ and meeting the condition that $S \in \mathcal{C}''$ if and only if all finite subsets of S belong to $\mathcal{C}''$. Also $\mathcal{C}''$ is U compatible if $\mathcal{C}'$ is.

Proof Again, we check each item of the definition:

0. If $\exists S \in \mathcal{C}''$ with an atomic formula and its conjugate, then the finite set of this two formulas belongs to $\mathcal{C}'$. This is a contradiction, since $\mathcal{C}'$ is an $\mathcal{A}$-consistency property. For the same reason, neither $F\top$ nor $T\bot$ are in S.

1. Suppose $S \in \mathcal{C}''$ and $\boldsymbol{\alpha} \in S$ and, by contradiction, $S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \notin \mathcal{C}''$. Hence there is a finite set $F_1 \subseteq S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$ such that $F_1 \notin \mathcal{C}'$. But $S \in \mathcal{C}''$, then all its finite subsets are in $\mathcal{C}'$, so F_1 cannot be a subset of S alone, hence $\boldsymbol{\alpha}_1$ or $\boldsymbol{\alpha}_2$ or both are in F_1. Now let F_1^1 be $F_1 - \{\boldsymbol{\alpha}_1\}$, F_1^2 be $F_1 - \{\boldsymbol{\alpha}_2\}$ and F_1^12 be $F_1 - \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$. Thus we have: F_1^1 is a finite set with $F_1^1 \subseteq S$ but $F_1^1 \cup \{\boldsymbol{\alpha}_1\} \notin \mathcal{C}'$ or F_1^2 is a finite set with $F_1^2 \subseteq S$ but $F_1^2 \cup \{\boldsymbol{\alpha}_2\} \notin \mathcal{C}'$ or F_1^12 is a finite set with $F_1^12 \subseteq S$ but $F_1^12 \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \notin \mathcal{C}'$ or all of them. Let $F = F_1^1 \cup F_1^2 \cup F_1^12 \cup \{\boldsymbol{\alpha}\}$. then F is finite and $F \subseteq S$ and since $S \in \mathcal{C}''$, $F \in \mathcal{C}'$. But $\mathcal{C}'$ is an $\mathcal{A}$-consistency property, and $\boldsymbol{\alpha} \in F$, hence $F \cup \{\boldsymbol{\alpha}_1\} \in \mathcal{C}'$, $F \cup \{\boldsymbol{\alpha}_2\} \in \mathcal{C}'$, and $F \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \in \mathcal{C}'$. Since $\mathcal{C}'$ is closed under subsets $F_1 \in \mathcal{C}'$, which is a contradiction. So $S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \in \mathcal{C}''$.

2. Suppose $S \in \mathcal{C}''$ and $\boldsymbol{\beta} \in S$ and, by contradiction, $S \cup \{\boldsymbol{\beta}_1\} \notin \mathcal{C}''$ and $S \cup \{\boldsymbol{\beta}_2\} \notin \mathcal{C}''$. Since $S \cup \{\boldsymbol{\beta}_1\} \notin \mathcal{C}''$ there is a finite subset $F_1 \subseteq S \cup \{\boldsymbol{\beta}_1\}$ such that $F_1 \notin \mathcal{C}'$. But $S \in \mathcal{C}''$ then, since all finite subsets of S are in $\mathcal{C}'$, F_1 cannot be a subset of S alone, hence $\boldsymbol{\beta}_1 \in F_1$. We write F_1^1 for $F_1 - \{\boldsymbol{\beta}_1\}$. Thus F_1^1 is a finite set and $F_1^1 \subseteq S$ but $F_1^1 \cup \{\boldsymbol{\beta}_1\} \notin \mathcal{C}'$. Similarly, $F_2 \subseteq S \cup \{\boldsymbol{\beta}_2\}$, F_2 finite and $F_2 \notin \mathcal{C}'$. Then we write F_2^1 for $F_2 - \{\boldsymbol{\beta}_2\}$ and conclude F_2^1 is finite, $F_2^1 \subseteq S$ but $F_2^1 \cup \{\boldsymbol{\beta}_2\} \notin \mathcal{C}'$.
   Let $F = F_1^1 \cup F_2^1 \cup \{\boldsymbol{\beta}\}$. F is finite and $F \subseteq S$ and $S \in \mathcal{C}''$, then $F \in \mathcal{C}'$ which is an $\mathcal{A}$-consistency property. But $\boldsymbol{\beta} \in F$, hence $F \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}'$ or $F \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}'$. But $\mathcal{C}'$ is closed under subsets, hence $F_1^1 \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}'$ or $F_2^1 \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}'$. Thus we have a contradiction. Hence $S \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}''$ or $S \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}''$.

3. Suppose $S \in \mathcal{C}''$ and $\pi^\square \in S$ and, by contradiction, $S^\square \cup \{\pi_0^\square\} \notin \mathcal{C}''$. Then there is some finite set $F_1 \subseteq S^\square \cup \{\pi_0^\square\} \notin \mathcal{C}'$. But F_1 cannot be a subset of $S^\square$ alone, since $S^\square \subseteq S$, and all finite subsets of S are in $\mathcal{C}'$. So $\pi_0^\square \in F_1$. Let $F_1' = F_1 - \{\pi_0^\square\}$. We have F_1' finite, $F_1' \subseteq S^\square \subseteq S$, but $F_1' \cup \{\pi_0^\square\} \notin \mathcal{C}'$. Let $F = F_1' \cup \{\pi^\square\}$. F is finite, $F \subseteq S$ and $S \in \mathcal{C}''$. So $F \in \mathcal{C}'$. But $\mathcal{C}'$ is an $\mathcal{A}$-consistency property, so $F^\square \cup \{\pi_0^\square\} \in \mathcal{C}'$. By construction we have $F^\square = F_1'^\square$, and since all elements of F_1' belong to $S^\square$ (again by construction), we have $F_1' = F_1'^\square$. Hence $F^\square = F_1'$. So, we have a contradiction. Hence $S^\square \cup \{\pi_0^\square\} \in \mathcal{C}''$.

4. Suppose $S \in \mathcal{C}''$, $\pi^\alpha \in S$ and, by contradiction, $S^\square \cup S^\alpha \cup \{\pi_0^\alpha\} \notin \mathcal{C}''$. Then there is some finite set $F \subseteq S^\square \cup S^\alpha \cup \{\pi_0^\alpha\} \notin \mathcal{C}'$. We consider two cases depending on where $\{\pi_0^\alpha\}$ is in F or not:

   i) $\{\pi_0^\alpha\} \notin F$. Then $F \subseteq S^\square \cup S^\alpha$, and F is finite. We construct $\boxed{\alpha}F = \{[\alpha]B : B \in (F \cap S^\alpha)\} \cup (F \cap S^\square)$. So we have a finite $\boxed{\alpha}F \subseteq S$, and $S \subseteq \mathcal{C}''$. So $\boxed{\alpha}F \in \mathcal{C}'$. But $\boxed{\alpha}F$ is finite, so $\boxed{\alpha}F \in \mathcal{C}''$. Since $\{\pi^\alpha\} \in S$ we have a finite $\boxed{\alpha}F \cup \{\pi^\alpha\} \in \mathcal{C}''$, so $\boxed{\alpha}F \cup \{\pi^\alpha\} \in \mathcal{C}'$, and since $\mathcal{C}'$ is an $\mathcal{A}$-consistency property, we have $F \cup \{\pi_0^\alpha\} \in \mathcal{C}'$. Finally, since $\mathcal{C}'$ is closed under subsets, we conclude $F \in \mathcal{C}'$, which is a contradiction.
   So we conclude $S^\square \cup S^\alpha \cup \{\pi_0^\alpha\} \in \mathcal{C}''$.

   ii) $\{\pi_0^\alpha\} \in F$. Then let F_1 be $F - \{\pi_0^\alpha\}$. We have F_1 finite, $F_1 \subseteq S^\square \cup S^\alpha$, so just as in the previous case, we conclude $F_1 \cup \{\pi_0^\alpha\} \in \mathcal{C}'$, which is a contradiction.

5. Suppose $S \in \mathcal{C}''$, $\mathcal{V}^\square \in S$ and, by contradiction, $S \cup \{\mathcal{V}_0^\square\} \notin \mathcal{C}''$. Then there is some finite set $F_1 \in S \cup \{\mathcal{V}_0^\square\} \notin \mathcal{C}'$. But F_1 can not be a subset of S alone, since all finite subsets of S are in $\mathcal{C}'$. So $\mathcal{V}_0^\square \in F_1$. Let $F_1' = F_1 - \{\mathcal{V}_0^\square\}$. We have F_1' finite, $F_1' \subseteq S$ but $F_1' \cup \{\mathcal{V}_0^\square\} \notin \mathcal{C}'$. Now, let $F = F_1' \cup \{\mathcal{V}^\square\}$. F is finite $F \subseteq S$ and $S \in \mathcal{C}''$, so $F \cup \{\mathcal{V}_0^\square\} \in \mathcal{C}'$. $\mathcal{C}'$ is closed under subsets, so it follows that $F_1' \cup \{\mathcal{V}_0^\square\} \in \mathcal{C}'$ and we have a contradiction. Hence, $S \cup \{\mathcal{V}_0^\square\} \in \mathcal{C}''$.

Since $\mathcal{C}'$ is closed under subsets, if $S \in \mathcal{C}'$, all finite subsets of S are in $\mathcal{C}'$, hence $S \in \mathcal{C}''$. Thus $\mathcal{C}''$ extends $\mathcal{C}'$.
Suppose F is finite. If $F \in \mathcal{C}''$ then all finite subsets of F must be in $\mathcal{C}'$, in particular $F \in \mathcal{C}'$. Conversely, if $F \in \mathcal{C}'$, by the previous paragraph, $F \in \mathcal{C}''$. Hence $\mathcal{C}'$ and $\mathcal{C}''$ agree on finite subsets. Now, $S \in \mathcal{C}''$ if and only if all finite subsets of S are in $\mathcal{C}'$ (by definition) if and only if all finite subsets of S are in $\mathcal{C}''$ (by the previous paragraph).

For the U-compatibility, suppose $S \in \mathcal{C}''$ and $Z \in U$. $S \in \mathcal{C}''$, so all finite subsets of S belongs to $\mathcal{C}'$. Let F finite and $F \subseteq S$. $F \in \mathcal{C}'$ and $\mathcal{C}'$ is U-compatible, hence $F \cup \{Z\} \in \mathcal{C}'$. Hence all finite subsets of $S \cup \{Z\}$ belongs to $\mathcal{C}''$ (by the previous paragraph). This concludes the proof.

Definition 5 A collection $\mathcal{C}$ of sets is said to be of finite character provided $S \in \mathcal{C}$ iff each finite subset of S belongs to $\mathcal{C}$.

Proposition 1 Any (U-compatible) $\mathcal{A}$-consistency property may be extended to a (U-compatible) $\mathcal{A}$-consistency property of finite character.

Proof This follows immediately from lemmas 3 and 4.

Proposition 2 Let $\mathcal{C}$ be an $\mathcal{A}$-consistency property of finite character. Each member of $\mathcal{C}$ may be extended to a maximal member of $\mathcal{C}$.

Proof The proof is identical as in [Fit83].

6.2 Model existence theorem for $\mathcal{A}$

In this section we present the proof of the model existence theorem of $\mathcal{A}$. This theorem will be used In the next section to show the completeness of $\mathcal{A}$.

Theorem 2 (Model existence theorem for $\mathcal{A}$) Let $\mathcal{C}$ be an $\mathcal{A}$-consistency property. If S is $\mathcal{C}$-consistent then S is $\mathcal{A}$-satisfiable.

Proof Let $\mathcal{C}$ be an $\mathcal{A}$-consistency property, and suppose S is $\mathcal{C}$-consistent ($S \in \mathcal{C}$). We show S is $\mathcal{A}$-satisfiable.
By proposition 1, $\mathcal{C} \subseteq \mathcal{C}'$, where $\mathcal{C}'$ is also an $\mathcal{A}$-consistency property and is of finite character. We use $\mathcal{C}'$ to create an $\mathcal{A}$-model. Let W be the set of all maximal members of $\mathcal{C}'$ (there are some, by proposition 2). Let ACT_0 be a set of atomic actions. For $v, w \in W$ let $vR_\alpha w$ mean $v^\alpha \subseteq w$ and $vR_\Box w$ mean $v^\Box \subseteq w$.

We note the following:
(1) $v \subseteq w \Rightarrow v^\Box \subseteq w^\Box$ and $v^\alpha \subseteq w^\alpha$.
(2) $v^\Box = v^{\Box\Box}$ and $v^\Box \subseteq v$.
We claim $(W, (R_\alpha)_{,\alpha \in ACT_0}, R_\Box)$ is a $\mathcal{A}$-frame. In fact, there is no restriction on R_α and since $v^\Box \subseteq v$, $R_\Box$ is reflexive. Now suppose $uR_\Box v$ and $vR_\Box w$, i.e., $u^\Box \subseteq v$ and $v^\Box \subseteq w$. Thus, by the above observation, $u^{\Box\Box} \subseteq v^\Box$. And since $u^\Box = u^{\Box\Box}$ we have $u^\Box \subseteq v^\Box \subseteq w$. Hence $u^\Box \subseteq w$, i.e., $uR_\Box w$. So $R_\Box$ is transitive. Hence $(W, (R_\alpha)_{,\alpha \in ACT}, R_\Box)$ is a $\mathcal{A}$-frame.
Now we define an interpretation v in $(W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box)$ as follows: for each $w \in W$ and for each propositional variable P, set

$$v(w, P) = \begin{cases} \text{T} & \text{if } TP \in w \\ \text{F} & \text{otherwise} \end{cases}$$

The interpretation v determines a unique model $\mu = (W, (R_\alpha)_{\alpha \in ACT_0}, R_\Box, \models)$ which is an $\mathcal{A}$-model. We claim: for each $w \in W$ and for each signed formula Z, $Z \in w$ implies $\models^\mu_w Z$. We prove this by induction on the size of Z:
Base step: If Z is of size 0, Z is TP or FP, where P is a variable. Then:
$TP \in w \Rightarrow v(w, P) = T$ (definitions of v). So $\models^\mu_w P$.
$FP \in w \Rightarrow TP \notin w$ ($\mathcal{C}'$ is an $\mathcal{A}$-consistency property so it does not contains an atomic formula and its conjugate). So $v(w, P) = F$ (definition of v). So $\not\models^\mu_w P$.
Induction Hypothesis (IH): Suppose $Z \in w$ implies $\models^\mu_w Z$ is valid for all $w \in W$ and all signed formulas of size less or equal than $n \geq 0$.

Suppose now Z is of size greater than $n > 0$. We have several cases depending on where Z is of type $\boldsymbol{\alpha}, \boldsymbol{\beta}, \pi^{\Box}, \pi^{\alpha}, \mathcal{V}^{\Box}, \mathcal{V}^{\alpha}$:

i) Case $Z = \boldsymbol{\alpha}$: then $\boldsymbol{\alpha} \in w$ implies $w \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \in \mathcal{C}'$ (since $\mathcal{C}'$ is a $\mathcal{A}$-consistency property). So $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in w$ (since w is maximal). Hence $\models^{\mu}_{w} \boldsymbol{\alpha}_1$ and $\models^{\mu}_{w} \boldsymbol{\alpha}_2$ (IH). So $\models^{\mu}_{w} \boldsymbol{\alpha}$.

ii) Case $Z = \boldsymbol{\beta}$: then $\boldsymbol{\beta} \in w$ implies (since $\mathcal{C}'$ is a $\mathcal{A}$-consistency property) $w \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}'$ or $w \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}'$. So $\boldsymbol{\beta}_1 \in w$ or $\boldsymbol{\beta}_2 \in w$ (since w is maximal). Hence $\models^{\mu}_{w} \boldsymbol{\beta}_1$ or $\models^{\mu}_{w} \boldsymbol{\beta}_2$ (IH). So $\models^{\mu}_{w} \boldsymbol{\beta}$.

iii) Case $Z = \pi^{\Box}$: then $\pi^{\Box} \in w$ implies $w^{\Box} \cup \{\pi^{\Box}_0\} \in \mathcal{C}'$ ($\mathcal{C}'$ is a $\mathcal{A}$-consistency property). Now let w' be a maximal extension in $\mathcal{C}'$ of $w^{\Box} \cup \{\pi^{\Box}_0\}$. Then $w' \in W$. Since $\pi^{\Box}_0 \in w'$ and $\pi^{\Box}_0$ is of lower size than $\pi^{\Box}$, by IH, $\models^{\mu}_{w'} \pi^{\Box}_0$. Since $w^{\Box} \subseteq w'$ we have $wR_{\Box}w'$. So $\models^{\mu}_{w} \pi^{\Box}$.

iv) Case $Z = \pi^{\alpha}$: then $\pi^{\alpha} \in w$ implies $w^{\alpha} \cup w^{\Box} \cup \{\pi^{\alpha}_0\} \in \mathcal{C}'$. Now let w' be a maximal extension in $\mathcal{C}'$ of $w^{\alpha} \cup w^{\Box} \cup \{\pi^{\alpha}_0\}$. Then $w' \in W$. Since $\pi^{\alpha}_0 \in w'$ and π^{α}_0 is of lower size than π^{α}, by IH $\models^{\mu}_{w'} \pi^{\alpha}_0$. But $w^{\alpha} \subseteq w'$, so $wR_{\alpha}w'$. Hence $\models^{\mu}_{w} \pi^{\alpha}$.

v) Case $Z = \mathcal{V}^{\Box}$: then $\mathcal{V}^{\Box} \in w$. Take any $w' \in W$ such that $wR_{\Box}w'$, that is, $w^{\Box} \subseteq w'$. We claim $\mathcal{V}^{\Box}_0 \in w'$. In fact, if $\mathcal{V}^{\Box} \in w$ then $\mathcal{V}^{\Box} \in w^{\Box}$, so $\mathcal{V}^{\Box} \in w'$. Since $\mathcal{V}^{\Box} \in w'$ we have $w' \cup \{\mathcal{V}^{\Box}_0\} \in \mathcal{C}'$ (and $\mathcal{C}'$ is an $\mathcal{A}$-consistency property). But w' is maximal in $\mathcal{C}'$, so $\mathcal{V}^{\Box}_0 \in w'$. Now $\mathcal{V}^{\Box}_0 \in w'$, then $\models^{\mu}_{w'} \mathcal{V}^{\Box}_0$ by IH. So we have $\forall w' \in W$ such that $wR_{\Box}w', \models^{\mu}_{w'} \mathcal{V}^{\Box}_0$. Hence, $\models^{\mu}_{w} \mathcal{V}^{\Box}$.

vi) Case $Z = \mathcal{V}^{\alpha}$: then $\mathcal{V}^{\alpha} \in w$. Take any $w' \in W$ such that $wR_{\alpha}w'$, that is, $w^{\alpha} \subseteq w'$. If $\mathcal{V}^{\alpha} \in w$ then by the definition of w^{α} we conclude $\mathcal{V}^{\alpha}_0 \in w^{\alpha}$. So $\mathcal{V}^{\alpha}_0 \in w'$. Hence by (IH), $\models^{\mu}_{w'} \mathcal{V}^{\alpha}_0$. Thus we have $\forall w' \in W$ such that $wR_{\alpha}w', \models^{\mu}_{w'} \mathcal{V}^{\alpha}_0$. Hence $\models^{\mu}_{w} \mathcal{V}^{\alpha}$.

To finish the proof, recall we started with an $\mathcal{A}$-consistency property $\mathcal{C}$ and a set $S \in \mathcal{C}$. Then $S \in \mathcal{C}'$. Extend S to a maximal member w of $\mathcal{C}'$ and by the above remarks w is $\mathcal{A}$-satisfiable, hence the subset S as well is.

Theorem 3 (Strong model existence theorem for $\mathcal{A}$) Let U be a set of signed formulas and let $\mathcal{C}$ be an $\mathcal{A}$-consistency property that is U-compatible. If S is $\mathcal{C}$-consistent, then S is satisfiable at a world in some $\mathcal{A}$-model $\mu = (W, (R_{\alpha})_{\alpha \in ACT_0}, R_{\Box}, \models)$ in which the members of U hold at every possible world.

Proof In addition to the previous proof, suppose $\mathcal{C}$ is U-compatible. Then so is $\mathcal{C}'$ (if w is a maximal member of $\mathcal{C}'$ then $U \subseteq w$, because $Z \in U$ then $w \cup \{Z\} \in \mathcal{C}'$ since $\mathcal{C}'$ is U-compatible). But w is maximal so $w \cup \{Z\} = w$. Hence $Z \in w$. Then $\models^{\mu}_{w} U$ for each $w \in W$. This completes the proof.

6.3 Completeness theorem of $\mathcal{A}$

In this section we prove the completeness theorem for $\mathcal{A}$. We will need to show a lemma.

Definition 6 Suppose $\{Z_1, \cdots, Z_n\}$ is a finite set of formulas. By an $\mathcal{A}$-tableau for $\{Z_1, \cdots, Z_n\}$ we mean any tableau that begins:

$$\begin{array}{c} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{array}$$

and continues by applications of the $\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{V}^{\Box}, \pi^{\Box}$ and π^{α}- rules.

Definition 7 A finite set S of signed formulas is $\mathcal{A}$-consistent if no $\mathcal{A}$-tableau for S closes.

Lemma 5 Let $\mathcal{C}$ be the collection of all $\mathcal{A}$-consistent sets. $\mathcal{C}$ is an $\mathcal{A}$-consistency property.

Proof We check each item of the definition:

0. Suppose $S \in \mathbb{C}$. If S contains an atomic formula and its conjugate then S is a closed tableau. So S is not $\mathcal{A}$-consistent, thus $S \notin \mathbb{C}$, which is a contradiction. So S does not contain a formula and its conjugate. For the same reason S does contain neither $F\top$ nor $T\bot$.

1. Suppose $S \in \mathbb{C}$, $\boldsymbol{\alpha} \in S$ and $S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \notin \mathbb{C}$. Then there is a closed tableau of the form:

$$\begin{array}{c} S \\ \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \theta \end{array}$$

   where θ represents the tableau structure below the occurrence of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. But since $\boldsymbol{\alpha} \in S$ we may begin an $\mathcal{A}$-tableau proof by an application of an $\boldsymbol{\alpha}$-rule, which results in:

$$\begin{array}{c} S \\ \boldsymbol{\alpha} \\ \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{array}$$

   Now we just copy the structure represented by θ and we get a closed tableau for S. So S is not $\mathcal{A}$-consistent, thus $S \notin \mathbb{C}$, which is a contradiction. Hence $S \cup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\} \in \mathbb{C}$.

2. Suppose $S \in \mathbb{C}$, $\boldsymbol{\beta} \in S$, and $S \cup \{\boldsymbol{\beta}_1\} \notin \mathbb{C}$ and $S \cup \{\boldsymbol{\beta}_2\} \notin \mathbb{C}$. If $S \cup \{\boldsymbol{\beta}_1\} \notin \mathbb{C}$ then there is a closed tableau of the form:

$$\begin{array}{c} S \\ \boldsymbol{\beta}_1 \\ \theta_1 \end{array}$$

   where θ_1 represents the tableau structure below the occurrence of $\boldsymbol{\beta}_1$. And if $S \cup \{\boldsymbol{\beta}_2\} \notin \mathbb{C}$ then there is a closed tableau of the form:

$$\begin{array}{c} S \\ \boldsymbol{\beta}_2 \\ \theta_2 \end{array}$$

   where θ_2 represents the tableau structure below the occurrence of $\boldsymbol{\beta}_2$. But $\boldsymbol{\beta} \in S$, so we may begin an $\mathcal{A}$-tableau proof by a $\boldsymbol{\beta}$-rule, like this:

$$\begin{array}{ccccc} & & S & & \\ & & \boldsymbol{\beta} & & \\ & \swarrow & & \searrow & \\ \boldsymbol{\beta}_1 & & & & \boldsymbol{\beta}_2 \end{array}$$

   and we can copy the structures θ_1 and θ_2 below the respective occurrences of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. In this way we obtain:

$$\begin{array}{ccccc} & & S & & \\ & & \boldsymbol{\beta} & & \\ & \swarrow & & \searrow & \\ \boldsymbol{\beta}_1 & & & & \boldsymbol{\beta}_2 \\ \theta_1 & & & & \theta_2 \end{array}$$

which is a closed tableau for S. So S is not $\mathcal{A}$-consistent, thus $S \notin \mathcal{C}$, which is a contradiction. Hence $S \cup \{\boldsymbol{\beta}_1\} \in \mathcal{C}$ or $S \cup \{\boldsymbol{\beta}_2\} \in \mathcal{C}$.

3. Suppose $S \in \mathcal{C}$, $\pi^\Box \in S$ and $S^\Box \cup \{\pi_0^\Box\} \notin \mathcal{C}$. Then there is a closed tableau of the form:

$$\begin{array}{c} S^\Box \\ \pi_0^\Box \\ \theta \end{array}$$

where θ represents the tableau structure below the occurrence of $\pi_0^\Box$. Now, since $\pi^\Box \in S$, we may begin an $\mathcal{A}$-tableau proof by a $\pi^\Box$-rule in S to obtain:

$$\begin{array}{c} S \\ \pi^\Box \\ \downarrow \\ S^\Box \\ \pi_0^\Box \end{array}$$

and if we copy the structure represented by θ we get a closed tableau for S. Thus S is not $\mathcal{A}$-consistent and so $S \notin \mathcal{C}$, which is a contradiction. Hence $S^\Box \cup \{\pi_0^\Box\} \in \mathcal{C}$.

4. Suppose $S \in \mathcal{C}$, $\pi^\alpha \in S$ and $S^\Box \cup S^\alpha \cup \{\pi_0^\alpha\} \notin \mathcal{C}$. Then there is a closed tableau of the form:

$$\begin{array}{c} S^\Box \\ S^\alpha \\ \pi_0^\alpha \\ \theta \end{array}$$

where θ represents the tableau structure below the occurrence of π_0^α. Now, since $\pi^\alpha \in S$, we may begin an $\mathcal{A}$-tableau proof by a π^α-rule in S to obtain:

$$\begin{array}{c} S \\ \pi^\alpha \\ \downarrow \\ S^\Box \\ S^\alpha \\ \pi_0^\alpha \end{array}$$

and if we copy the structure represented by θ we get a closed tableau for S. Thus S is not $\mathcal{A}$-consistent and so $S \notin \mathcal{C}$, which is a contradiction. Hence $S^\Box \cup S^\alpha \cup \{\pi_0^\Box\} \in \mathcal{C}$.

5. Suppose $S \in \mathcal{C}$, $\mathcal{V}^\Box \in S$ and $S \cup \{\mathcal{V}_0^\Box\} \notin \mathcal{C}$. Then there is a closed tableau of the form:

$$\begin{array}{c} S \\ \mathcal{V}_0^\Box \\ \theta \end{array}$$

where θ represents the tableau structure below the occurrence of $\mathcal{V}_0^\Box$. Now, since $\mathcal{V}^\Box \in S$, we may begin an $\mathcal{A}$-tableau proof by a $\mathcal{V}^\Box$-rule in S to obtain:

$$\begin{array}{c} S \\ \mathcal{V}^\Box \\ \mathcal{V}_0^\Box \end{array}$$

and if we copy the structure represented by θ we get a closed tableau for S. Thus S is not $\mathcal{A}$-consistent and so $S \notin \mathcal{C}$, which is a contradiction. Hence $S \cup \{\mathcal{V}_0^\Box\} \in \mathcal{C}$.

This finishes the proof of the lemma.

Theorem 4 (Completeness theorem for $\mathcal{A}$) Let X be a formula. If $\models_{\mathcal{A}} X$ then X has an $\mathcal{A}$-tableau proof.

Proof Suppose A has no $\mathcal{A}$-tableau proof. Then there is no closed tableau for $\{FA\}$. Hence by lemma 5, $\{FA\} \in \mathcal{C}$. By the model existence theorem, $\{FA\}$ is $\mathcal{A}$-satisfiable, hence A is not valid in all $\mathcal{A}$-models. This is a contradiction.

7 Conclusion

We have presented the definition of the logic $\mathcal{A}$, which is a version of Propositional Dynamic Logic (PDL) that we claim suitable for reasoning about actions and plans. This logic is an alternative to PDL in the sense that it presents an alternative to the iteration operator. It is a formal system with "almost" the same expressive power of PDL, in the sense that we are able to speak about atomic and compound actions, as well as action laws, integrity constraints, goals and the existence of plans. We have presented a tableau system for this logic in order to allow automatic deductions of facts. We have showed correctness and completeness results. We have presented open problems which are subjetc of future research.

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