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Relatório Técnico
RT-DINF 002/2004

Curitiba, PR
2004

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Abstract

Let G be a simple and finite graph. In this paper we are concerned with operations on G that transform it into a perfect graph. We define some graph parameters related to these operations and prove some results about it. Using a well know lower bound for Ramsey Numbers we conclude that there are graphs that are highly imperfect.

Keywords: perfect graphs, perfect subgraphs, perfect supergraphs, operation on graphs, vertex deletion, edge deletion, edge insertion, edge editing.

1 Preliminaries

All graphs in this paper are finite and simple. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$. We use $\chi(G)$, $\alpha(G)$ and $\omega(G)$ for the chromatic number, the independence number and the clique number of a graph G respectively. The complement of a graph G is denoted by $\overline{G}$. When there is no ambiguity we use χ and $\overline{\chi}$ in the place of $\chi(G)$ and $\chi(\overline{G})$ respectively. The same interpretation should be made of α, $\overline{\alpha}$, ω and $\overline{\omega}$.

Given a graph G we denote by $G - A$ the graph obtained from the deletion of a set A of edges (or vertices with the care to take only induced subgraphs) from $E(G)$ (resp. $V(G)$). Similarly we use $G + A$ in the case of the insertion of a set A of edges (vertices) into $E(G)$ (resp. $V(G)$). A *hole* in G is a induced cycle in G of length at least 4. An *antihole* in G is a induced subgraph of G whose complement is a hole. We say that a hole (antihole) with an odd number of vertices is an *odd* hole (antihole).

Definition 1.1 (Perfect Graphs) *A graph G is perfect if, for all induced subgraphs H of G, the identity $\chi(H) = \omega(H)$ holds.*

Theorem 1.1 (Perfect Graph Theorem) *A graph G is perfect if and only if $\overline{G}$ is perfect.*

Theorem 1.2 (Strong Perfect Graph Theorem) *A graph G is perfect if and only if G it contains neither odd holes nor odd antiholes as subgraphs.*

The proof of theorem 1.1 can be found in [Diest00]. The second theorem was conjectured by Berge in 1961 and was settled recently in a joint work of Chudnovsky, Robertson, Seymour and Thomas [Chudn03].

The *Ramsey number* $r(k,l)$ is the smallest integer such that for every graph G with $|V(G)| \geq r(k,l)|$ it holds that $\alpha(G) \geq k$ and $\omega(G) \geq l$. The Ramsey Theorem [Bondy76] states that $r(k,l)$ is well defined for all positive integers k and l. A graph G with $|V(G)| = r(k,l) - 1$ and with $\alpha(G) < k$ and $\omega(G) < l$ is called a *$r(k,l)$-ramsey graph.*

A well know lower bound for $r(k,l)$ is presented in the following theorem [Bondy76]:

Theorem 1.3 *Let $m = min(k,l)$, then it holds that $r(k,l) \geq 2^m$.*

2 Operations to make a Graph Perfect

Now we define four graph parameters: ρ_1, ρ_2, ρ_3 and ρ_4. Each one of these parameters is related to an operation that can be performed on a graph to make it perfect.

Definition 2.1 (Maximum Perfect Subgraph) *Given a graph G, we denote by $\rho_1(G)$ the size of the smallest set $A \subseteq E(G)$, such that $G - A$ is perfect.*

Definition 2.2 (Minimum Perfect Completion) *Given a graph G, we denote by $\rho_2(G)$ the size of the smallest edge set B, where $B \cap E(G) = \varnothing$, such that $G + B$ is perfect.*

Definition 2.3 (Closest Perfect Subgraph) *Given a graph G, we denote by $\rho_3(G)$ the size of the smallest integer $|A| + |B|$, such that $A \subseteq E(G)$ and B is an edge set with $B \cap E(G) = \varnothing$, and $(G - A) + B$ is perfect.*

Definition 2.4 (Maximum Induced Perfect Subgraph) *Given a graph G, we denote by $\rho_4(G)$ the size of the smallest set $X \subseteq V(G)$, such that $G - X$ is perfect.*

It is worth observing that in the definitions 2.1, 2.3 and 2.4 we are interested in perfect subgraphs and in definition 2.2 we are interested in perfect supergraphs.

When there is no ambiguity we write only ρ_1 in the place of $\rho_1(G)$ and $\overline{\rho_1}$ in the place of $\rho_1(\overline{G})$. Similarly we use ρ_2, $\overline{\rho_2}$, ρ_3, $\overline{\rho_3}$, ρ_4 and $\overline{\rho_4}$.

Fact 2.1 *For all graphs it holds that:*

i. $\rho_4 = \overline{\rho_4}$.

ii. $\rho_1 = \overline{\rho_2}$ *and* $\rho_2 = \overline{\rho_1}$.

iii. $\rho_3 = \overline{\rho_3}$.

Proof: *(i)* It comes directly from theorem 1.1. *(ii)* Let G be a graph with $\rho_1(G) = k$ and let $A \subseteq E(G)$ be a set of k edges such that $G - A$ is perfect. It follows from theorem 1.1 that $\overline{G} + A$ is perfect and that obviously A is minimal. The identity $\rho_2 = \overline{\rho_1}$ comes from the observation that if $\rho_1 = \overline{\rho_2}$ then $\overline{\rho_1} = \overline{\overline{\rho_2}} = \rho_2$. *(iii)* Let $A \subseteq E(G)$ and $B \cap E(G) = \varnothing$ such that $\rho_3(G) = |A| + |B|$ and $(G - A) + B$ is perfect. We know from theorem 1.1 (again) that $(\overline{G} - B) + A$ is perfect and that $|A| + |B|$ is minimum to make $\overline{G}$ a perfect graph. □

Theorem 2.1 *For all graphs it holds that $\rho_4 \leq \rho_3 \leq \rho_1, \rho_2$.*

Proof: The inequality $\rho_3 \leq \rho_1, \rho_2$ comes directly from the inequality $\rho_3 \leq min(\rho_1, \rho_2)$. Now we need to show that $\rho_4 \leq \rho_3$. Let G be a graph with $\rho_3(G) = k$. We can show that $\rho_4 \leq \rho_3$ exhibiting a set $X \subseteq V(G)$, with $|X| \leq k$, such that $G - X$ is perfect.

Let $A = \{a_1, a_2, ..., a_{k_1}\} \subseteq E(G)$ and $B = \{b_1, b_2, ..., b_{k_2}\}$, where $b_i \notin E(G)$ for $(1 \leq i \leq k_2)$, such that $H = (G - A) + B$ is perfect and $|A| + |B| = k$. Let $a_i = \{v^1_{a_i}, v^2_{a_i}\}$ for $1 \leq a_i \leq k_1$ and $b_i = \{v^1_{b_i}, v^2_{b_i}\}$ for $1 \leq b_i \leq k_2$. Take $X = X_1 \cup X_2$ where the vertex sets $X_1 = \{x^1_1, x^2_1, ..., x^{k_1}_1\}$ and $X_2 = \{x^1_2, x^2_2, ..., x^{k_1}_2\}$ are built in this manner:

For $1 \leq i \leq k_1$, let each $x^i_1 = v^j_{a_i}$, for $j = 1$ or $j = 2$. For $1 \leq i \leq k_2$, let each $x^i_2 = v^j_{b_i}$, for $j = 1$ or $j = 2$. In other words, the vertices to be removed from G to make it perfect are vertices that are adjacent to the $\rho_3(G)$ edges that should be removed from and inserted in G to make it perfect.

Now we show that for X built in this manner we it holds that $G - X$ is perfect. Suppose that $G - X$ is imperfect. Then there are an odd hole either in $G - X$ or in $\overline{G} - X$, with, let us say the vertices $u_1, u_2, ..., u_l$, for some odd l, $5 \leq l \leq |V(G)| - |X|$. If this odd hole is present in $G - X$ it must also be present in H. If this odd hole is present in $\overline{G} - X$ it must also be present in $\overline{H}$. But H (and $\overline{H}$), is perfect, what is a contradiction. □

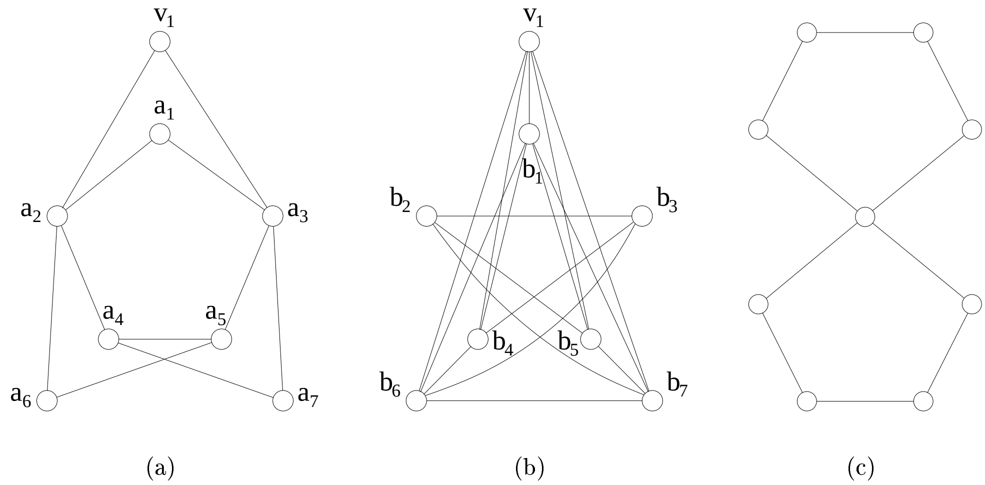

Figure 1: (a) The graph G. (b) The graph H (isomorphic to $\overline{G}$). (c) A graph with $\rho_4 < \rho_3$.

It is important to note that the definition of ρ_3 do not have much information if there are no graphs for which the strict inequality $\rho_3 < \rho_1, \rho_2$ holds. We can contruct such graphs from the graphs G and H presented in figures 1(a) e 1(b) respectively. The constructed graph appears in figure 2. Now, we see this construction in detail.

First of all, note that $\rho_1(G) = 2$ and $\rho_2(G) = 1$ (the insertion of the edge a_2a_3, for example, makes the graph perfect). As a consequence we have $\rho_3(G) = 1$. For the graph H we have $\rho_1(H) = 2$ and $\rho_2(H) = \rho_3(H) = 1$ (the deletion of the edge b_2b_3, for example, makes the graph perfect), since it is isomorphic to $\overline{G}$.

Let G' with vertex set $V(G') = V(G) \cup V(H)$ and edge set $E(G') = E(G) \cup E(H)$ (see figure 2). Removing the edge b_2b_3 from $E(G')$ and inserting a_2a_3 in it, we get a perfect graph. By the construction of G' it is easy to see that $G'' = (G' + \{a_2a_3\}) - \{b_2b_3\}$ does not have odd holes.

Now we have to look at $\overline{G''}$. By the construction we know that the subgraph of $\overline{G'}$ induced by $\{v_1, a_1, a_2, ..., a_7\}$ does not have odd holes. The same occurs for the subgraph induced by $\{v_1, b_1, b_2, ..., b_7\}$. Let $A = \{a_1, ..., a_7\}$ and $B = \{b_1, ..., b_7\}$.

The last thing to check is if there is an odd hole using vertices from both A and B. But such holes does not exists in $\overline{G''}$ since the subgraph of $\overline{G''}$ induced by $A \cup B$ is the complete bipartite graph $K_{7,7}$ (the two independent sets are A and B).

We also have a case where strict inequality $\rho_4 < \rho_3$ occurs. An example is showed in figure 1(c).

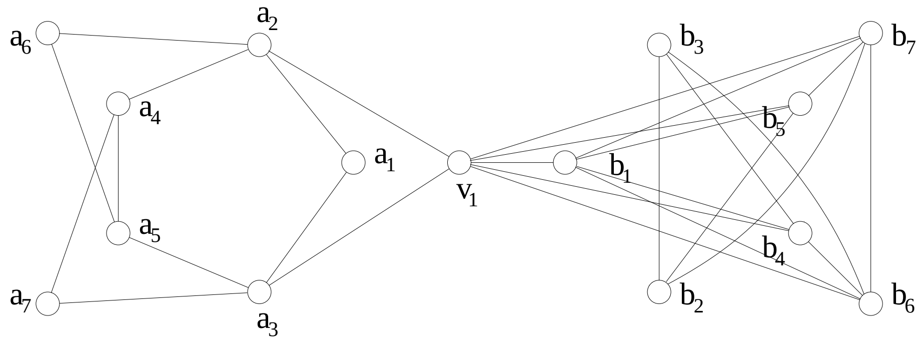

Figure 2: The graph G'.

3 Lower Bounds and Imperfect Graphs

We start this section with some lower bounds for ρ_4.

FACT 3.1 *For all graphs with n vertices we have*

i. $\chi - \omega \leq \rho_4$

ii. $\overline{\chi} - \alpha \leq \rho_4$

iii. $n/\alpha - \omega \leq \rho_4$

PROOF: *(i)* This inequality is a consequence that for all $v \in V(G)$ we have $\chi(G-\{v\}) \geq \chi(G) - 1$ (in other words the chromatic number can not decay more than one with a vertex deletion). *(ii)* It comes directely from (i) and from fact 2.1(ii). *(iii)* This inequality comes from (i) and from the well know inequality $\alpha\chi \geq n$. □

From the theorem 2.1, we have that the bounds from Lema 3.1 are valid for ρ_1, ρ_2 and ρ_3.

Since there are graphs with arbitrary high chromatic number and with clique number $\omega = 2$ [Bondy76], we can conclude from lemma 3.1(i) that there are graphs with arbitrarily high ρ_4. However, from this result we can not know if ρ_4 is high when compared to the size of the vertex set of the graph. We can gain more information about it from lemma 3.1(iii) combinated with the theorem 1.3. From these results we know that there exist graphs with high ρ_4 even when compared to the size of the vertex set. In some way we can say that these graphs are highly imperfect. We state this result in the following way:

THEOREM 3.1 *There exist graphs with n vertices and*

$$\rho_4 \geq \frac{n}{\lg(2n)} - \lg(2n).$$

PROOF: From the Ramsey Theorem we know that $r(k,k)$ is well defined for any integer $k > 0$. From theorem 1.3 we have $r(k,k) \geq 2^{k/2}$. Let G be a (k,k)-ramsey graph. In other words, G is a graph with $r(k,k)-1$ vertices, with $\alpha(G) < k$ and $\omega(G) < k$. From lema 3.1(iii) we have

$$\rho_4 \geq \frac{r(k,k)-1}{(k-1)} - (k-1) \geq \frac{n}{\lg(2n)} - \lg(2n). \quad \square$$

4 Remarks

4.1 Upper Bounds

Two obvious upper bounds for ρ_1 and ρ_2 (and consequentely for ρ_3 and ρ_4) are the following:

FACT 4.1 *For all graphs with n vertices and m edges we have*

i. $\rho_1 \leq min\left(m - (n-1), \frac{m}{2}\right)$

ii. $\rho_2 \leq min\left(\left[\binom{n}{2} - m\right] - (n-1), \frac{\binom{n}{2} - m}{2}\right)$

PROOF: *(i)* It comes directly from the fact that trees and bipartite graphs are perfect and from the fact that for any graph G, there exists $X \subseteq E(G)$, $|X| \leq m/2$, such that $G - X$ is bipartite (see [Alon92]). *(ii)* It follows from the fact 2.1(ii). □

4.2 Computational Complexity

Since perfectness is a hereditary (on the induced subgraphs) and non trivial property, it follows from the result of Lewis and Yannakakis [Lewis80] (stated below) that the problem of finding ρ_4 is a NP-complete problem.

THEOREM 4.1 (LEWIS AND YANNAKAKIS) *If Π is a grapy property satisfying the following conditions:*

i. There are infinitely many graphs for whitch Π holds.

ii. There are infinitely many graphs for whitch Π does not holds.

iii. If Π holds for a graph G then Π holds for all induced subgraphs of G.

Then the following problem is NP-complete: Given a graph G and a positive integer $k \leq |V(G)|$, is there a subset $V' \subseteq V(G)$ with $|V'| \geq k$ such that Π holds for the subgraph of G induced by V'?

Natanzon, Shamir and Sharan [Natan99] showed in 1999 that the decision version of the problems of finding a maximum perfect subgraph, a minimum perfect completion and a closest perfect subgraph [1] are also NP-complete. These three problems can be seen as the same as the problem of findind ρ_1, ρ_2 and ρ_3.

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