Matemática Discreta: Material Complementar

Renato Carmo

Introdução

Este documento recolhe material complementar às aulas da disciplina **Matemática Discreta (CI1237)** do **Deparatamento de Informática da UFPR**.

1 Uma Prova do Princípio da Inclusão Exclusão

Teorema (Princípio da Inclusão–Exclusão). *Se A_1, ..., A_n são conjuntos finitos, então*

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

Demonstração. Vamos provar que

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|,$$

para todo $n \in \mathbb{N}$ por indução em n.

H.I.: Seja $a \in \mathbb{N}$ tal que, dados conjuntos finitos $A_1, \ldots, A_p$,

$$\left| \bigcup_{k=1}^{p} A_k \right| = \sum_{k=1}^{p} (-1)^{k+1} \sum_{I \in \binom{[p]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

para todo $p \in [0..a]$.

Passo: Vamos provar que

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \sum_{k=1}^{a+1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

Inicialmente,

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \left| \left(\bigcup_{k=1}^{a} A_k \right) \cup A_{a+1} \right|$$

$$\overset{\text{C. ??}}{=} \left| \bigcup_{k=1}^{a} A_k \right| + |A_{a+1}| - \left| \left(\bigcup_{k=1}^{a} A_k \right) \cap A_{a+1} \right|$$

$$= \mathcal{A} + |A_{a+1}| + \mathcal{B},$$

onde

$$\mathcal{A} := \left| \bigcup_{k=1}^{a} A_k \right|,$$

$$\mathcal{B} := -\left| \left(\bigcup_{k=1}^{a} A_k \right) \cap A_{a+1} \right|.$$

$$\mathcal{A} = \left| \bigcup_{k=1}^{a} A_k \right| \overset{\text{HI}}{=} \sum_{k=1}^{a} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

$$= \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + \sum_{k=2}^{a} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

$$= \mathcal{C} + \mathcal{D},$$

onde

$$\mathcal{C} := \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right|,$$

$$\mathcal{D} := \sum_{k=2}^{a} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

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Então
$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{A} + |A_{a+1}| + \mathcal{B} = \mathcal{C} + \mathcal{D} + |A_{a+1}| + \mathcal{B},$$

Como

$$\mathcal{C} + |A_{a+1}| = \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + |A_{a+1}|$$

$$= (-1)^{1+1} \sum_{I \in \binom{[a]}{1}} \left| \bigcap_{i \in I} A_i \right| + |A_{a+1}| = \sum_{i \in [a]} |A_i| + |A_{a+1}|$$

$$= \sum_{i \in [a+1]} |A_i| = (-1)^{1+1} \sum_{I \in \binom{[a+1]}{1}} \left| \bigcap_{i \in I} A_i \right| = \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

$$= \mathcal{E},$$

onde

$$\mathcal{E} := \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

e

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{E} + \mathcal{D} + \mathcal{B},$$

Temos

$$\mathcal{B} = -\left| \left(\bigcup_{k=1}^{a} A_k \right) \cap A_{a+1} \right|.$$

Como

$$\left(\bigcup_{k=1}^{a} A_k \right) \cap A_{a+1} \stackrel{\text{Ex. ??}}{=} \bigcup_{k=1}^{a} (A_k \cap A_{a+1}) = \bigcup_{k=1}^{a} B_k,$$

onde

$$B_k := A_k \cap A_{a+1}, \ \text{para todo } k \in [1..a],$$

então

$$\mathcal{B} = -\left| \bigcup_{k=1}^{a} B_k \right| \stackrel{\text{HI}}{=} -\sum_{k=1}^{a} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} B_i \right| = \sum_{k=1}^{a} (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} B_i \right|$$

Como para cada $I \subseteq [a]$ temos

$$\bigcap_{i \in I} B_i = \bigcap_{i \in I}(A_i \cap A_{a+1}) = \left(\bigcap_{i \in I} A_i\right) \cap A_{a+1} = \bigcap_{i \in I \cup \{a+1\}} A_i,$$

então

$$\begin{aligned}
\mathcal{B} &= \sum_{k=1}^{a}(-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \\
&= \sum_{k=1}^{a-1}(-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| + \sum_{k=a}^{a}(-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \\
&= \mathcal{F} + \mathcal{G},
\end{aligned}$$

onde

$$\begin{aligned}
\mathcal{F} &:= \sum_{k=1}^{a-1}(-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|, \\
\mathcal{G} &:= \sum_{k=a}^{a}(-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|.
\end{aligned}$$

de forma que

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{E} + \mathcal{D} + \mathcal{B} = \mathcal{E} + \mathcal{D} + \mathcal{F} + \mathcal{G}.$$

$$\mathcal{F} = \sum_{k=1}^{a-1}(-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| = \sum_{k=2}^{a}(-1)^{k+1} \sum_{I \in \binom{[a]}{k-1}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|,$$

$$\mathcal{D} + \mathcal{F} = \sum_{k=2}^{a}(-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + \sum_{k=2}^{a}(-1)^{k+1} \sum_{I \in \binom{[a]}{k-1}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|$$

$$= \sum_{k=2}^{a}(-1)^{k+1} \left(\sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + \sum_{I \in \binom{[a]}{k-1}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \right)$$

$$\stackrel{\text{T. }\mathbf{??}}{=} \sum_{k=2}^{a}(-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

$$= \mathcal{H}$$

de forma que

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{E} + \mathcal{D} + \mathcal{F} + \mathcal{G} = \mathcal{E} + \mathcal{H} + \mathcal{G}.$$

$$\mathcal{G} = \sum_{k=a}^{a}(-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|$$

$$= (-1)^{a+2} \sum_{I \in \binom{[a]}{a}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| = (-1)^{a+2} \left| \bigcap_{i \in [a] \cup \{a+1\}} A_i \right|$$

$$= (-1)^{a+2} \left| \bigcap_{i \in [a+1]} A_i \right| = (-1)^{a+2} \sum_{I \in \binom{[a+1]}{a+1}} \left| \bigcap_{i \in I} A_i \right|$$

$$= \sum_{k=a+1}^{a+1}(-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

Finalmente,

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{E} + \mathcal{H} + \mathcal{G}$$

$$= \sum_{k=1}^{1}(-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

$$+ \sum_{k=2}^{a}(-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

$$+ \sum_{k=a+1}^{a+1}(-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

$$= \sum_{k=1}^{a+1}(-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

Base: Vamos provar que

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{I \subseteq [n]}(-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|,$$

para todo $n \in [0..2]$.

$n = 0$: temos

$$\left| \bigcup_{k=1}^{0} A_k \right| = |\emptyset| = 0,$$

e

$$\sum_{I \subseteq [0]}(-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = \sum_{I \subseteq \emptyset}(-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = 0.$$

$n = 1$: temos

$$\left| \bigcup_{k=1}^{1} A_k \right| = |A_1|,$$

e

$$\sum_{I \subseteq [1]}(-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = (-1)^{|\emptyset|+1} \left| \bigcap_{i \in \emptyset} A_i \right| + (-1)^{|[1]|+1} \left| \bigcap_{i \in [1]} A_i \right|$$

$$= (-1)^{0+1} |\emptyset| + (-1)^{1+1} |A_1| = -1 \times 0 + 1 \times |A_1|$$

$$= |A_1|.$$

$n = 2$: temos
$$\left| \bigcup_{k=1}^{2} A_k \right| = |A_1 \cup A_2|,$$

e

$$\sum_{I \subseteq [2]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = (-1)^{|\emptyset|+1} \left| \bigcap_{i \in \emptyset} A_i \right|$$

$$+ (-1)^{|\{1\}|+1} \left| \bigcap_{i \in \{1\}} A_i \right| + (-1)^{|\{2\}|+1} \left| \bigcap_{i \in \{2\}} A_i \right|$$

$$+ (-1)^{|\{1,2\}|+1} \left| \bigcap_{i \in \{1,2\}} A_i \right|$$

$$= (-1)^{0+1} |\emptyset| + (-1)^{1+1} |A_1| + (-1)^{1+1} |A_2| + (-1)^{2+1} |A_1 \cap A_2|$$

$$= -1 \times 0 + 1 \times |A_1| + 1 \times |A_2| + -1 \times |A_1 \cap A_2|$$

$$= |A_1| + |A_2| - |A_1 \cap A_2|.$$

que é verdade pelo Corolário **??**.

$\square$