Matemática Discreta

Unidade 11: Exercícios de Indução (2)

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Exercício 52 (Fibonacci)

Exercício 52

A **sequência de Fibonacci** é a função $F\colon \mathbb{N} \to \mathbb{N}$ dada por

$$F(n) = \begin{cases} n, & \text{se } n \leq 1 \\ F(n-1) + F(n-2), & \text{se } n > 1. \end{cases}$$

Exercício 52

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$$F(n) = \begin{cases} n, & \text{se } n \leq 1 \\ F(n-1) + F(n-2), & \text{se } n > 1. \end{cases}$$

(a) Prove por indução em n que

$$F(n) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \text{ para todo } n \in \mathbb{N}$$

Exercício 52.(a)

Vamos provar que

$$F(n) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right),$$

por indução em n.

HI: Seja $a \in \mathbb{N}$ tal que

$$F(k) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right), \text{ para todo } k \in [0..a],$$

Exercício 52.(a)

Passo: Vamos provar que

$$F(a+1) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a+1}\right).$$

Exercício 52.(a)

Passo: Vamos provar que

$$F(a+1) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a+1}\right).$$

Pela definição de F temos que para todo $a > 1$,

$$F(a+1) = F(a) + F(a-1).$$

Exercício 52.(a)

Passo: Vamos provar que

$$F(a+1) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a+1}\right).$$

Pela definição de F temos que para todo $a > 1$,

$$F(a+1) = F(a) + F(a-1).$$

Como $a \in [0..a]$, temos da HI que

$$F(a) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} - \left(\frac{1-\sqrt{5}}{2}\right)^{a}\right).$$

Exercício 52.(a)

Passo: Vamos provar que

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Como $a \in [0..a]$, temos da HI que

$$F(a) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} - \left(\frac{1-\sqrt{5}}{2}\right)^{a}\right).$$

Como $a-1 \in [0..a]$, temos da HI que

$$F(a-1) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\right).$$

Exercício 52.(a)

Passo: ...
Então

$$F(a+1) = F(a) + F(a-1)$$

Exercício 52.(a)

Passo: ...

Então

$$
\begin{aligned}
F(a+1) &= F(a) + F(a-1) \\
&= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} - \left(\frac{1-\sqrt{5}}{2}\right)^{a}\right) + \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\right)
\end{aligned}
$$

Exercício 52.(a)

Passo: ...

Então

$$
\begin{aligned}
F(a+1) &= F(a) + F(a-1) \\
&= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} - \left(\frac{1-\sqrt{5}}{2}\right)^{a}\right) + \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\right) \\
&= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} - \left(\frac{1-\sqrt{5}}{2}\right)^{a} + \left(\frac{1+\sqrt{5}}{2}\right)^{a-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\right)
\end{aligned}
$$

Exercício 52.(a)

Passo: ...

Então

$$
\begin{aligned}
F(a+1) &= F(a) + F(a-1) \\
&= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} - \left(\frac{1-\sqrt{5}}{2}\right)^{a}\right) + \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\right) \\
&= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} - \left(\frac{1-\sqrt{5}}{2}\right)^{a} + \left(\frac{1+\sqrt{5}}{2}\right)^{a-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\right) \\
&= \frac{\sqrt{5}}{5}\left(\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a} + \left(\frac{1+\sqrt{5}}{2}\right)^{a-1}\right) - \left(\left(\frac{1-\sqrt{5}}{2}\right)^{a} + \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\right)\right)
\end{aligned}
$$

Exercício 52.(a)

Passo: ...

$$F(a+1)=\frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1}\left(1+\left(\frac{1+\sqrt{5}}{2}\right)\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\left(1+\left(\frac{1-\sqrt{5}}{2}\right)\right)\right)$$

Exercício 52.(a)

Passo: ...

$$F(a+1) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1}\left(1+\left(\frac{1+\sqrt{5}}{2}\right)\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\left(1+\left(\frac{1-\sqrt{5}}{2}\right)\right)\right)$$

$$= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right)$$

Exercício 52.(a)

Passo: ...

$$
\begin{aligned}
F(a+1) &= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1}\left(1+\left(\frac{1+\sqrt{5}}{2}\right)\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\left(1+\left(\frac{1-\sqrt{5}}{2}\right)\right)\right)\\
&= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2}-\left(\frac{1-\sqrt{5}}{2}\right)^{a-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right)\\
&= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{a+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{a+1}\right).
\end{aligned}
$$

Exercício 52.(a)

Base: Vamos provar que

$$F(b) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^b - \left(\frac{1-\sqrt{5}}{2}\right)^b\right), \text{ para todo } b \in \{0, 1\}$$

Exercício 52.(a)

Base: Vamos provar que

$$F(b) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^b - \left(\frac{1-\sqrt{5}}{2}\right)^b\right), \text{ para todo } b \in \{0, 1\}$$

isto é, vamos provar que

$$\begin{aligned} F(0) &= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right) \\ F(1) &= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right). \end{aligned}$$

Exercício 52.(a)

Base: ...

Por um lado,

$$\begin{aligned} F(0) &= 0, \\ F(1) &= 1. \end{aligned}$$

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Por outro lado,

$$\frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right) = \frac{\sqrt{5}}{5}(1-1) = \frac{\sqrt{5}}{5}(0) = 0,$$

Exercício 52.(a)

Base: ...

Por um lado,

$$\begin{aligned} F(0) &= 0, \\ F(1) &= 1. \end{aligned}$$

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e

$$\frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) = \frac{\sqrt{5}}{5}\left(\frac{2\sqrt{5}}{2}\right)$$

Exercício 52.(a)

Base: ...

Por um lado,

$$\begin{aligned} F(0) &= 0, \\ F(1) &= 1. \end{aligned}$$

Por outro lado,

$$\frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right) = \frac{\sqrt{5}}{5}(1-1) = \frac{\sqrt{5}}{5}(0) = 0,$$

e

$$\frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) = \frac{\sqrt{5}}{5}\left(\frac{2\sqrt{5}}{2}\right) = \frac{5}{5} = 1.$$

Exercício 52.(a)

Base: ...

Portanto,

$$
\begin{aligned}
F(0) &= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{0} - \left(\frac{1-\sqrt{5}}{2}\right)^{0}\right) \\
F(1) &= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{1} - \left(\frac{1-\sqrt{5}}{2}\right)^{1}\right),
\end{aligned}
$$

Exercício 52.(a)

Base: ...

Portanto,

$$
\begin{aligned}
F(0) &= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right)\\
F(1) &= \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right),
\end{aligned}
$$

Portanto,

$$
F(n) = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \text{ para todo } n \in \mathbb{N},
$$

Exercício 52.(b)

A **sequência de Fibonacci** é a função $F\colon \mathbb{N} \to \mathbb{N}$ dada por

$$F(n) = \begin{cases} n, & \text{se } n \leq 1 \\ F(n-1) + F(n-2), & \text{se } n > 1. \end{cases}$$

(b) Conclua que

$$F(n) \approx \frac{\sqrt{5}}{5} \left(\frac{1+\sqrt{5}}{2} \right)^n$$

Exercício 52.(b)

Como

$$\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \overset{\text{Ex. 19}}{\approx} \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

Exercício 52.(b)

Como

$$\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \overset{\text{Ex. 19}}{\approx} \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

então

$$F(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}(1+\epsilon(n))\right)^n,$$

para alguma função $\epsilon\colon \mathbb{N} \to \mathbb{R}$ tal que $\lim \epsilon(n) = 0$.

Exercício 52.(b)

Então,

$$F(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n (1+\epsilon(n))^n \,.$$

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Como

$$\lim \epsilon(n) = 0,$$

então

$$\lim(1+\epsilon(n)) = 1,$$

Exercício 52.(b)

Então,

$$F(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n (1+\epsilon(n))^n.$$

Como

$$\lim \epsilon(n) = 0,$$

então

$$\lim(1+\epsilon(n)) = 1,$$

e daí,

$$\lim(1+\epsilon(n))^n = 1,$$

Exercício 52.(b)

Então,

$$F(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n (1+\epsilon(n))^n\,.$$

Como

$$\lim \epsilon(n) = 0,$$

então

$$\lim(1+\epsilon(n)) = 1,$$

e daí,

$$\lim(1+\epsilon(n))^n = 1,$$

e, consequentemente,

$$F(n) \approx \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$