

Matemática Discreta

Unidade 71: O Princípio da Inclusão-Exclusão

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Segundo Período Especial de 2020

União de Conjuntos

Corolário 48

União de Conjuntos

Corolário 48: $|A_1 \cup A_2|$

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2|$

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

caso geral

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

caso geral: $|A_1 \cup \dots \cup A_n|$

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

caso geral: $|A_1 \cup \dots \cup A_n| = ???$

União de 3 Conjuntos

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$$|A_1 \cup A_2 \cup A_3|$$

União de 3 Conjuntos

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$

União de 3 Conjuntos

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \end{aligned}$$

União de 3 Conjuntos

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &\quad + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

União de 4 Conjuntos

União de 4 Conjuntos

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4|$$

União de 4 Conjuntos

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|) \end{aligned}$$

União de 4 Conjuntos

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &\quad + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \end{aligned}$$

União de 4 Conjuntos

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &\quad + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\ &\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

Caso Geral

$$|A_1 \cup \dots \cup A_n|$$

Caso Geral

$$|A_1 \cup \dots \cup A_n| = (|A_1| + \dots + |A_n|)$$

Caso Geral

$$|A_1 \cup \dots \cup A_n| = (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|)$$

Caso Geral

$$|A_1 \cup \dots \cup A_n| = (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots$$

Caso Geral

$$|A_1 \cup \dots \cup A_n| = (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n|$$

Caso Geral

$$\begin{aligned} |A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| \end{aligned}$$

Caso Geral

$$\begin{aligned} |A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| \end{aligned}$$

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$$\begin{aligned}|A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \dots\end{aligned}$$

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$$\begin{aligned} |A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^{n+1} \sum_{\{i_1, \dots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \dots \cap A_{i_n}| \end{aligned}$$

Caso Geral

$$\begin{aligned}|A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^{n+1} \sum_{\{i_1, \dots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \dots \cap A_{i_n}| \\&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right|\end{aligned}$$

Caso Geral

$$\begin{aligned}|A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^{n+1} \sum_{\{i_1, \dots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \dots \cap A_{i_n}| \\&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right|\end{aligned}$$

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$$\begin{aligned}|A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^{n+1} \sum_{\{i_1, \dots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \dots \cap A_{i_n}| \\&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right| - \dots\end{aligned}$$

Caso Geral

$$\begin{aligned} |A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^{n+1} \sum_{\{i_1, \dots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \dots \cap A_{i_n}| \\ &= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right| - \dots + (-1)^{n+1} \sum_{I \in \binom{[n]}{n}} \left| \bigcap_{i \in I} A_i \right| \end{aligned}$$

Caso Geral

$$\begin{aligned}|A_1 \cup \dots \cup A_n| &= (|A_1| + \dots + |A_n|) - (|A_1 \cap A_2| + \dots + |A_{n-1} \cap A_n|) - \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n| \\&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^{n+1} \sum_{\{i_1, \dots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \dots \cap A_{i_n}| \\&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right| - \dots + (-1)^{n+1} \sum_{I \in \binom{[n]}{n}} \left| \bigcap_{i \in I} A_i \right| \\&= \sum_{k=1}^n \left((-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left| \bigcap_{i \in I} A_i \right| \right).\end{aligned}$$

Teorema 80 (Princípio da Inclusão–Exclusão)

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$$\left| \bigcup_{k=1}^n A_k \right| = \sum_{k=1}^n (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

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$$\left| \bigcup_{k=1}^n A_k \right| = \sum_{k=1}^n (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

Demonstração.

Indução em n

