Matemática Discreta

Unidade 71: O Princípio da Inclusão-Exclusão

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União de Conjuntos

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Corolário 48

União de Conjuntos

Corolário 48: $|A_1 \cup A_2|$

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2|$

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

caso geral

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

caso geral: $|A_1 \cup \ldots \cup A_n|$

União de Conjuntos

Corolário 48: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

caso geral: $|A_1 \cup \ldots \cup A_n| =$???

União de 3 Conjuntos

União de 3 Conjuntos

$$|A_1 \cup A_2 \cup A_3|$$

União de 3 Conjuntos

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$

União de 3 Conjuntos

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)$$

União de 3 Conjuntos

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3| = & |A_1| + |A_2| + |A_3| \\
& - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\
& + |A_1 \cap A_2 \cap A_3|
\end{aligned}
$$

União de 4 Conjuntos

União de 4 Conjuntos

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4|$$

União de 4 Conjuntos

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| \\ - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|)$$

União de 4 Conjuntos

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\
& - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\
& + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|)
\end{aligned}
$$

União de 4 Conjuntos

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\
& - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\
& + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\
& - |A_1 \cap A_2 \cap A_3 \cap A_4|
\end{aligned}
$$

Caso Geral

Caso Geral

$|A_1 \cup \ldots \cup A_n|$

Caso Geral

$$|A_1 \cup \ldots \cup A_n| = (|A_1| + \ldots + |A_n|)$$

Caso Geral

$$|A_1 \cup \ldots \cup A_n| = (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|)$$

Caso Geral

$$|A_1 \cup \ldots \cup A_n| = (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) - \ldots$$

Caso Geral

$$|A_1 \cup \ldots \cup A_n| = (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n$$

Caso Geral

$$|A_1 \cup \ldots \cup A_n| = (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n$$
$$= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}|$$

Caso Geral

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n \\
&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}|
\end{aligned}
$$

Caso Geral

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&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \ldots
\end{aligned}
$$

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\begin{aligned}
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&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \ldots + (-1)^{n+1} \sum_{\{i_1, \ldots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \ldots \cap A_{i_n}|
\end{aligned}
$$

Caso Geral

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\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n \\
&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \ldots + (-1)^{n+1} \sum_{\{i_1, \ldots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \ldots \cap A_{i_n}| \\
&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right|
\end{aligned}
$$

Caso Geral

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) - \ldots + (-1)^{n+1}|A_1 \cap \ldots \cap A_n \\
&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \ldots + (-1)^{n+1} \sum_{\{i_1, \ldots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \ldots \cap A_{i_n}| \\
&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right|
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$$

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\begin{aligned}
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&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \ldots + (-1)^{n+1} \sum_{\{i_1, \ldots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \ldots \cap A_{i_n}| \\
&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right| - \ldots
\end{aligned}
$$

Caso Geral

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= (|A_1| + \ldots + |A_n|) - (|A_1 \cap A_2| + \ldots + |A_{n-1} \cap A_n|) - \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n| \\
&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \ldots + (-1)^{n+1} \sum_{\{i_1, \ldots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \ldots \cap A_{i_n}| \\
&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right| - \ldots + (-1)^{n+1} \sum_{I \in \binom{[n]}{n}} \left| \bigcap_{i \in I} A_i \right|
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&= \sum_{\{i_1\} \in \binom{[n]}{1}} |A_{i_1}| - \sum_{\{i_1, i_2\} \in \binom{[n]}{2}} |A_{i_1} \cap A_{i_2}| - \ldots + (-1)^{n+1} \sum_{\{i_1, \ldots, i_n\} \in \binom{[n]}{n}} |A_{i_1} \cap \ldots \cap A_{i_n}| \\
&= \sum_{I \in \binom{[n]}{1}} \left| \bigcap_{i \in I} A_i \right| - \sum_{I \in \binom{[n]}{2}} \left| \bigcap_{i \in I} A_i \right| - \ldots + (-1)^{n+1} \sum_{I \in \binom{[n]}{n}} \left| \bigcap_{i \in I} A_i \right| \\
&= \sum_{k=1}^{n} \left((-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left| \bigcap_{i \in I} A_i \right| \right).
\end{aligned}
$$

Teorema 80 (Princípio da Inclusão–Exclusão)

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$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left|\bigcap_{i \in I} A_i\right|$$

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$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left|\bigcap_{i \in I} A_i\right|$$

Demonstração.

Indução em n □