

# Matemática Discreta: Material Complementar

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## Introdução

Este documento recolhe material complementar às aulas da disciplina Matemática Discreta (CI237) do Departamento de Informática da UFPR.

## 1 Uma Prova do Princípio da Inclusão Exclusão

**Teorema** (Princípio da Inclusão–Exclusão). *Se  $A_1, \dots, A_n$  são conjuntos finitos, então*

$$\left| \bigcup_{k=1}^n A_k \right| = \sum_{k=1}^n (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

*Demonstração.* Vamos provar que

$$\left| \bigcup_{k=1}^n A_k \right| = \sum_{I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|,$$

para todo  $n \in \mathbb{N}$  por indução em  $n$ .

**H.I.:** Seja  $a \in \mathbb{N}$  tal que, dados conjuntos finitos  $A_1, \dots, A_p$ ,

$$\left| \bigcup_{k=1}^p A_k \right| = \sum_{k=1}^p (-1)^{k+1} \sum_{I \in \binom{[p]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

para todo  $p \in [0..a]$ .

**Passo:** Vamos provar que

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \sum_{k=1}^{a+1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

Inicialmente,

$$\begin{aligned} \left| \bigcup_{k=1}^{a+1} A_k \right| &= \left| \left( \bigcup_{k=1}^a A_k \right) \cup A_{a+1} \right| \\ &\stackrel{\text{C. ??}}{=} \left| \bigcup_{k=1}^a A_k \right| + |A_{a+1}| - \left| \left( \bigcup_{k=1}^a A_k \right) \cap A_{a+1} \right| \\ &= \mathcal{A} + |A_{a+1}| + \mathcal{B}, \end{aligned}$$

onde

$$\begin{aligned} \mathcal{A} &:= \left| \bigcup_{k=1}^a A_k \right|, \\ \mathcal{B} &:= - \left| \left( \bigcup_{k=1}^a A_k \right) \cap A_{a+1} \right|. \end{aligned}$$


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$$\begin{aligned} \mathcal{A} &= \left| \bigcup_{k=1}^a A_k \right| \stackrel{\text{HI}}{=} \sum_{k=1}^a (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| \\ &= \sum_{k=1}^1 (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + \sum_{k=2}^a (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| \\ &= \mathcal{C} + \mathcal{D}, \end{aligned}$$

onde

$$\begin{aligned} \mathcal{C} &:= \sum_{k=1}^1 (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right|, \\ \mathcal{D} &:= \sum_{k=2}^a (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| \end{aligned}$$

Então

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{A} + |A_{a+1}| + \mathcal{B} = \mathcal{C} + \mathcal{D} + |A_{a+1}| + \mathcal{B},$$


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Como

$$\begin{aligned} \mathcal{C} + |A_{a+1}| &= \sum_{k=1}^1 (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + |A_{a+1}| \\ &= (-1)^{1+1} \sum_{I \in \binom{[a]}{1}} \left| \bigcap_{i \in I} A_i \right| + |A_{a+1}| = \sum_{i \in [a]} |A_i| + |A_{a+1}| \\ &= \sum_{i \in [a+1]} |A_i| = (-1)^{1+1} \sum_{I \in \binom{[a+1]}{1}} \left| \bigcap_{i \in I} A_i \right| = \sum_{k=1}^1 (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right| \\ &= \mathcal{E}, \end{aligned}$$

onde

$$\mathcal{E} := \sum_{k=1}^1 (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|$$

e

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{E} + \mathcal{D} + \mathcal{B},$$


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Temos

$$\mathcal{B} = - \left| \left( \bigcup_{k=1}^a A_k \right) \cap A_{a+1} \right|.$$

Como

$$\left( \bigcup_{k=1}^a A_k \right) \cap A_{a+1} \stackrel{\text{Ex.??}}{=} \bigcup_{k=1}^a (A_k \cap A_{a+1}) = \bigcup_{k=1}^a B_k,$$

onde

$$B_k := A_k \cap A_{a+1}, \text{ para todo } k \in [1..a],$$

então

$$\mathcal{B} = - \left| \bigcup_{k=1}^a B_k \right| \stackrel{\text{HI}}{=} - \sum_{k=1}^a (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} B_i \right| = \sum_{k=1}^a (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} B_i \right|$$

Como para cada  $I \subseteq [a]$  temos

$$\bigcap_{i \in I} B_i = \bigcap_{i \in I} (A_i \cap A_{a+1}) = \left( \bigcap_{i \in I} A_i \right) \cap A_{a+1} = \bigcap_{i \in I \cup \{a+1\}} A_i,$$

então

$$\begin{aligned} \mathcal{B} &= \sum_{k=1}^a (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \\ &= \sum_{k=1}^{a-1} (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| + \sum_{k=a}^a (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \\ &= \mathcal{F} + \mathcal{G}, \end{aligned}$$

onde

$$\begin{aligned} \mathcal{F} &:= \sum_{k=1}^{a-1} (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|, \\ \mathcal{G} &:= \sum_{k=a}^a (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|. \end{aligned}$$

de forma que

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{E} + \mathcal{D} + \mathcal{B} = \mathcal{E} + \mathcal{D} + \mathcal{F} + \mathcal{G}.$$

$$\mathcal{F} = \sum_{k=1}^{a-1} (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| = \sum_{k=2}^a (-1)^{k+1} \sum_{I \in \binom{[a]}{k-1}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right|,$$

$$\begin{aligned}
\mathcal{D} + \mathcal{F} &= \sum_{k=2}^a (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + \sum_{k=2}^a (-1)^{k+1} \sum_{I \in \binom{[a]}{k-1}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \\
&= \sum_{k=2}^a (-1)^{k+1} \left( \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I} A_i \right| + \sum_{I \in \binom{[a]}{k-1}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \right) \\
&\stackrel{\text{T. ??}}{=} \sum_{k=2}^a (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right| \\
&= \mathcal{H}
\end{aligned}$$

de forma que

$$\left| \bigcup_{k=1}^{a+1} A_k \right| = \mathcal{E} + \mathcal{D} + \mathcal{F} + \mathcal{G} = \mathcal{E} + \mathcal{H} + \mathcal{G}.$$


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$$\begin{aligned}
\mathcal{G} &= \sum_{k=a}^a (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| \\
&= (-1)^{a+2} \sum_{I \in \binom{[a]}{a}} \left| \bigcap_{i \in I \cup \{a+1\}} A_i \right| = (-1)^{a+2} \left| \bigcap_{i \in [a] \cup \{a+1\}} A_i \right| \\
&= (-1)^{a+2} \left| \bigcap_{i \in [a+1]} A_i \right| = (-1)^{a+2} \sum_{I \in \binom{[a+1]}{a+1}} \left| \bigcap_{i \in I} A_i \right| \\
&= \sum_{k=a+1}^{a+1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|
\end{aligned}$$


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Finalmente,

$$\begin{aligned}
\left| \bigcup_{k=1}^{a+1} A_k \right| &= \mathcal{E} + \mathcal{H} + \mathcal{G} \\
&= \sum_{k=1}^1 (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right| \\
&\quad + \sum_{k=2}^a (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right| \\
&\quad + \sum_{k=a+1}^{a+1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right| \\
&= \sum_{k=1}^{a+1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left| \bigcap_{i \in I} A_i \right|
\end{aligned}$$

**Base:** Vamos provar que

$$\left| \bigcup_{k=1}^n A_k \right| = \sum_{I \subseteq [n]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|,$$

para todo  $n \in [0..2]$ .

$n = 0$ : temos

$$\left| \bigcup_{k=1}^0 A_k \right| = |\emptyset| = 0,$$

e

$$\sum_{I \subseteq [0]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = \sum_{I \subseteq \emptyset} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| = 0.$$

$n = 1$ : temos

$$\left| \bigcup_{k=1}^1 A_k \right| = |A_1|,$$

e

$$\begin{aligned}
\sum_{I \subseteq [1]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| &= (-1)^{|\emptyset|+1} \left| \bigcap_{i \in \emptyset} A_i \right| + (-1)^{|[1]|+1} \left| \bigcap_{i \in [1]} A_i \right| \\
&= (-1)^{0+1} |\emptyset| + (-1)^{1+1} |A_1| = -1 \times 0 + 1 \times |A_1| \\
&= |A_1|.
\end{aligned}$$

$n = 2$ : temos

$$\left| \bigcup_{k=1}^2 A_k \right| = |A_1 \cup A_2|,$$

e

$$\begin{aligned} \sum_{I \subseteq [2]} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| &= (-1)^{|\emptyset|+1} \left| \bigcap_{i \in \emptyset} A_i \right| \\ &+ (-1)^{|\{1\}|+1} \left| \bigcap_{i \in \{1\}} A_i \right| + (-1)^{|\{2\}|+1} \left| \bigcap_{i \in \{2\}} A_i \right| \\ &+ (-1)^{|\{1,2\}|+1} \left| \bigcap_{i \in \{1,2\}} A_i \right| \\ &= (-1)^{0+1} |\emptyset| + (-1)^{1+1} |A_1| + (-1)^{1+1} |A_2| + (-1)^{2+1} |A_1 \cap A_2| \\ &= -1 \times 0 + 1 \times |A_1| + 1 \times |A_2| + -1 \times |A_1 \cap A_2| \\ &= |A_1| + |A_2| - |A_1 \cap A_2|. \end{aligned}$$

que é verdade pelo Corolário ??.

□