Matemática Discreta: Material Complementar

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Introdução

Este documento recolhe material complementar às aulas da disciplina Matemática Discreta (CI237) do Deparatamento de Informática da UFPR.

1 Uma Prova do Princípio da Inclusão Exclusão

Teorema (Princípio da Inclusão–Exclusão)**.** *Se $A_1, \ldots, A_n$ são conjuntos finitos, então*

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{I\in\binom{[n]}{k}} \left|\bigcap_{i\in I} A_i\right|$$

Demonstração. Vamos provar que

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{I\subseteq[n]}(-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right|,$$

para todo $n \in \mathbb{N}$ por indução em n.

H.I.: Seja $a \in \mathbb{N}$ tal que, dados conjuntos finitos $A_1, \ldots, A_p$,

$$\left|\bigcup_{k=1}^{p} A_k\right| = \sum_{k=1}^{p}(-1)^{k+1} \sum_{I\in\binom{[p]}{k}} \left|\bigcap_{i\in I} A_i\right|$$

para todo $p \in [0..a]$.

Passo: Vamos provar que

$$\left|\bigcup_{k=1}^{a+1} A_k\right| = \sum_{k=1}^{a+1}(-1)^{k+1} \sum_{I\in\binom{[a+1]}{k}} \left|\bigcap_{i\in I} A_i\right|$$

Inicialmente,

$$\begin{aligned}\left|\bigcup_{k=1}^{a+1} A_k\right| &= \left|\left(\bigcup_{k=1}^{a} A_k\right) \cup A_{a+1}\right| \\ &\overset{\text{C. III.9}}{=} \left|\bigcup_{k=1}^{a} A_k\right| + |A_{a+1}| - \left|\left(\bigcup_{k=1}^{a} A_k\right) \cap A_{a+1}\right| \\ &= \mathcal{A} + |A_{a+1}| + \mathcal{B},\end{aligned}$$

onde

$$\begin{aligned}\mathcal{A} &:= \left|\bigcup_{k=1}^{a} A_k\right|, \\ \mathcal{B} &:= -\left|\left(\bigcup_{k=1}^{a} A_k\right) \cap A_{a+1}\right|.\end{aligned}$$

$$\begin{aligned}\mathcal{A} = \left|\bigcup_{k=1}^{a} A_k\right| &\overset{\text{HI}}{=} \sum_{k=1}^{a}(-1)^{k+1} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I} A_i\right| \\ &= \sum_{k=1}^{1}(-1)^{k+1} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I} A_i\right| + \sum_{k=2}^{a}(-1)^{k+1} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I} A_i\right| \\ &= \mathcal{C} + \mathcal{D},\end{aligned}$$

onde

$$\begin{aligned}\mathcal{C} &:= \sum_{k=1}^{1}(-1)^{k+1} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I} A_i\right|, \\ \mathcal{D} &:= \sum_{k=2}^{a}(-1)^{k+1} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I} A_i\right|\end{aligned}$$

Então
$$\left|\bigcup_{k=1}^{a+1} A_k\right| = \mathcal{A} + |A_{a+1}| + \mathcal{B} = \mathcal{C} + \mathcal{D} + |A_{a+1}| + \mathcal{B},$$

Como
$$\begin{aligned}
\mathcal{C} + |A_{a+1}| &= \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left|\bigcap_{i \in I} A_i\right| + |A_{a+1}| \\
&= (-1)^{1+1} \sum_{I \in \binom{[a]}{1}} \left|\bigcap_{i \in I} A_i\right| + |A_{a+1}| = \sum_{i \in [a]} |A_i| + |A_{a+1}| \\
&= \sum_{i \in [a+1]} |A_i| = (-1)^{1+1} \sum_{I \in \binom{[a+1]}{1}} \left|\bigcap_{i \in I} A_i\right| = \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left|\bigcap_{i \in I} A_i\right| \\
&= \mathcal{E},
\end{aligned}$$
onde
$$\mathcal{E} := \sum_{k=1}^{1} (-1)^{k+1} \sum_{I \in \binom{[a+1]}{k}} \left|\bigcap_{i \in I} A_i\right|$$
e
$$\left|\bigcup_{k=1}^{a+1} A_k\right| = \mathcal{E} + \mathcal{D} + \mathcal{B},$$

Temos
$$\mathcal{B} = -\left|\left(\bigcup_{k=1}^{a} A_k\right) \cap A_{a+1}\right|.$$
Como
$$\left(\bigcup_{k=1}^{a} A_k\right) \cap A_{a+1} \overset{\text{Ex. 52}}{=} \bigcup_{k=1}^{a} (A_k \cap A_{a+1}) = \bigcup_{k=1}^{a} B_k,$$
onde
$$B_k := A_k \cap A_{a+1}, \text{ para todo } k \in [1..a],$$
então
$$\mathcal{B} = -\left|\bigcup_{k=1}^{a} B_k\right| \overset{\text{HI}}{=} -\sum_{k=1}^{a} (-1)^{k+1} \sum_{I \in \binom{[a]}{k}} \left|\bigcap_{i \in I} B_i\right| = \sum_{k=1}^{a} (-1)^{k+2} \sum_{I \in \binom{[a]}{k}} \left|\bigcap_{i \in I} B_i\right|$$

Como para cada $I \subseteq [a]$ temos

$$\bigcap_{i\in I} B_i = \bigcap_{i\in I}(A_i \cap A_{a+1}) = \left(\bigcap_{i\in I} A_i\right) \cap A_{a+1} = \bigcap_{i\in I\cup\{a+1\}} A_i,$$

então

$$\begin{aligned}
\mathcal{B} &= \sum_{k=1}^{a}(-1)^{k+2} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I\cup\{a+1\}} A_i\right| \\
&= \sum_{k=1}^{a-1}(-1)^{k+2} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I\cup\{a+1\}} A_i\right| + \sum_{k=a}^{a}(-1)^{k+2} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I\cup\{a+1\}} A_i\right| \\
&= \mathcal{F} + \mathcal{G},
\end{aligned}$$

onde

$$\begin{aligned}
\mathcal{F} &:= \sum_{k=1}^{a-1}(-1)^{k+2} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I\cup\{a+1\}} A_i\right|, \\
\mathcal{G} &:= \sum_{k=a}^{a}(-1)^{k+2} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I\cup\{a+1\}} A_i\right|.
\end{aligned}$$

de forma que

$$\left|\bigcup_{k=1}^{a+1} A_k\right| = \mathcal{E} + \mathcal{D} + \mathcal{B} = \mathcal{E} + \mathcal{D} + \mathcal{F} + \mathcal{G}.$$

$$\mathcal{F} = \sum_{k=1}^{a-1}(-1)^{k+2} \sum_{I\in\binom{[a]}{k}} \left|\bigcap_{i\in I\cup\{a+1\}} A_i\right| = \sum_{k=2}^{a}(-1)^{k+1} \sum_{I\in\binom{[a]}{k-1}} \left|\bigcap_{i\in I\cup\{a+1\}} A_i\right|,$$

$$\begin{aligned}
\mathcal{D}+\mathcal{F} &= \sum_{k=2}^{a}(-1)^{k+1}\sum_{I\in\binom{[a]}{k}}\left|\bigcap_{i\in I}A_i\right| + \sum_{k=2}^{a}(-1)^{k+1}\sum_{I\in\binom{[a]}{k-1}}\left|\bigcap_{i\in I\cup\{a+1\}}A_i\right| \\
&= \sum_{k=2}^{a}(-1)^{k+1}\left(\sum_{I\in\binom{[a]}{k}}\left|\bigcap_{i\in I}A_i\right| + \sum_{I\in\binom{[a]}{k-1}}\left|\bigcap_{i\in I\cup\{a+1\}}A_i\right|\right) \\
&\overset{\text{T. III.35}}{=} \sum_{k=2}^{a}(-1)^{k+1}\sum_{I\in\binom{[a+1]}{k}}\left|\bigcap_{i\in I}A_i\right| \\
&= \mathcal{H}
\end{aligned}$$

de forma que

$$\left|\bigcup_{k=1}^{a+1}A_k\right| = \mathcal{E}+\mathcal{D}+\mathcal{F}+\mathcal{G} = \mathcal{E}+\mathcal{H}+\mathcal{G}.$$

$$\begin{aligned}
\mathcal{G} &= \sum_{k=a}^{a}(-1)^{k+2}\sum_{I\in\binom{[a]}{k}}\left|\bigcap_{i\in I\cup\{a+1\}}A_i\right| \\
&= (-1)^{a+2}\sum_{I\in\binom{[a]}{a}}\left|\bigcap_{i\in I\cup\{a+1\}}A_i\right| = (-1)^{a+2}\left|\bigcap_{i\in[a]\cup\{a+1\}}A_i\right| \\
&= (-1)^{a+2}\left|\bigcap_{i\in[a+1]}A_i\right| = (-1)^{a+2}\sum_{I\in\binom{[a+1]}{a+1}}\left|\bigcap_{i\in I}A_i\right| \\
&= \sum_{k=a+1}^{a+1}(-1)^{k+1}\sum_{I\in\binom{[a+1]}{k}}\left|\bigcap_{i\in I}A_i\right|
\end{aligned}$$

Finalmente,

$$\begin{aligned}
\left|\bigcup_{k=1}^{a+1} A_k\right| &= \mathcal{E} + \mathcal{H} + \mathcal{G} \\
&= \sum_{k=1}^{1}(-1)^{k+1} \sum_{I\in\binom{[a+1]}{k}} \left|\bigcap_{i\in I} A_i\right| \\
&\quad + \sum_{k=2}^{a}(-1)^{k+1} \sum_{I\in\binom{[a+1]}{k}} \left|\bigcap_{i\in I} A_i\right| \\
&\quad + \sum_{k=a+1}^{a+1}(-1)^{k+1} \sum_{I\in\binom{[a+1]}{k}} \left|\bigcap_{i\in I} A_i\right| \\
&= \sum_{k=1}^{a+1}(-1)^{k+1} \sum_{I\in\binom{[a+1]}{k}} \left|\bigcap_{i\in I} A_i\right|
\end{aligned}$$

Base: Vamos provar que

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{I\subseteq[n]} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right|,$$

para todo $n \in [0..2]$.

$n = 0$**:** temos

$$\left|\bigcup_{k=1}^{0} A_k\right| = |\emptyset| = 0,$$

e

$$\sum_{I\subseteq[0]} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right| = \sum_{I\subseteq\emptyset} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right| = 0.$$

$n = 1$**:** temos

$$\left|\bigcup_{k=1}^{1} A_k\right| = |A_1|,$$

e

$$\begin{aligned}
\sum_{I\subseteq[1]} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right| &= (-1)^{|\emptyset|+1} \left|\bigcap_{i\in\emptyset} A_i\right| + (-1)^{|[1]|+1} \left|\bigcap_{i\in[1]} A_i\right| \\
&= (-1)^{0+1} |\emptyset| + (-1)^{1+1} |A_1| = -1 \times 0 + 1 \times |A_1| \\
&= |A_1|.
\end{aligned}$$

$n = 2$**:** temos

$$\left|\bigcup_{k=1}^{2} A_k\right| = |A_1 \cup A_2|,$$

e

$$\begin{aligned}\sum_{I\subseteq[2]} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right| &= (-1)^{|\emptyset|+1} \left|\bigcap_{i\in\emptyset} A_i\right| \\ &\quad + (-1)^{|\{1\}|+1} \left|\bigcap_{i\in\{1\}} A_i\right| + (-1)^{|\{2\}|+1} \left|\bigcap_{i\in\{2\}} A_i\right| \\ &\quad + (-1)^{|\{1,2\}|+1} \left|\bigcap_{i\in\{1,2\}} A_i\right| \\ &= (-1)^{0+1} |\emptyset| + (-1)^{1+1} |A_1| + (-1)^{1+1} |A_2| + (-1)^{2+1} |A_1 \cap A_2| \\ &= -1 \times 0 + 1 \times |A_1| + 1 \times |A_2| + -1 \times |A_1 \cap A_2| \\ &= |A_1| + |A_2| - |A_1 \cap A_2|.\end{aligned}$$

que é verdade pelo Corolário III.9.

□