Otimização

Exercícios

15 de dezembro de 2020

1 Introdução / Problemas

1. (Papadimitriou and Steiglitz, 1998, Chap. 1, Ex. 1)

   Formulate the following as optimization problem instances, giving in each case the domain of feasible solutions F and the cost function c.

   (a) Find the shortest path between two nodes in a graph with edge weights representing distance.

   (b) Solve the *Tower of Hanoi* problem, which is defined as follows: We have 3 diamond needles and 64 gold disks of increasing diameter. The disks have holes at their centers, so that they fit on the needles, and initially the disks are all on the first needle, with the smallest on top, the next smallest below that, and so on, with the largest at the bottom. A legal move is the transfer of the top disk from any needle to any other, *with the condition that no disk is ever placed on top of one smaller than itself.* The problem is to transfer, by a sequence of legal moves, all the disks from the first to the second needle. (There is a story that the world will end when this task is completed [Kr], which may be an optimistic expectation.) Generalize to n gold disks.

   (c) Win a game of chess. How many instances of this problem are there?

   (d) Find a cylinder with a given surface area A that has the largest volume V.

   (e) Find a closed plane curve of given perimeter that encloses the largest area.

2 PL: Introdução e Modelagem

2. (Dasgupta, Papadimitriou, and Vazirani, 2006, ex. 7.1)

   Consider the following linear program

   max $5x + 3y$
   s.t.
   $5x - 2y \geq 0$
   $x + y \leq 7$
   $x \leq 5$
   $x \geq 0$
   $y \geq 0$

   Plot the feasible region and identify the optimal solution

3. (Dasgupta et al., 2006, ex. 7.3)

   A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.

   - Material 1 has density 2 tons/cubic meter, maximum available amount 40 cubic meters, and revenue $\$1,000$ per cubic meter.
   - Material 2 has density 1 ton/cubic meter, maximum available amount 30 cubic meters, and revenue $\$1,200$ per cubic meter.
   - Material 3 has density 3 tons/cubic meter, maximum available amount 20 cubic meters, and revenue $\$12,000$ per cubic meter.

   Write a linear program that optimizes revenue within the constraints.

4. (Dasgupta et al., 2006, ex. 7.4)

   Moe is deciding how much Regular Duff beer and how much Duff Strong beer to order each week. Regular Duff costs Moe \$1 per pint and he sells it at \$2 per pint; Duff Strong costs Moe \$1.50 per pint and he sells it at \$3 per pint. However, as part of a complicated marketing scam, the Duff company will only sell a pint of Duff Strong for each two pints or more of Regular Duff that Moe buys. Furthermore, due to past events that are better left untold, Duff will not sell Moe more than 3,000 pints per week. Moe knows that he can sell however much beer he has. Formulate a linear program for deciding how much Regular Duff and how much Duff Strong to buy, so as to maximize Moe's profit. Solve the program geometrically.

5. (Dasgupta et al., 2006, ex. 7.5)

   The Canine Products company offers two dog foods, Frisky Pup and Husky Hound, that are made from a blend of cereal and meat. A package of Frisky Pup requires 1 pound of cereal and 1.5 pounds of meat, and sells for $7. A package of Husky Hound uses 2 pounds of cereal and 1 pound of meat, and sells for $6. Raw cereal costs $1 per pound and raw meat costs $2 per pound. It also costs $1.40 to package the Frisky Pup and $0.60 to package the Husky Hound. A total of $240,000$ pounds of cereal and $180,000$ pounds of meat are available each month. The only production bottleneck is that the factory can only package $110,000$ bags of Frisky Pup per month. Needless to say, management would like to maximize profit.

   (a) Formulate the problem as a linear program in two variables.

   (b) Graph the feasible region, give the coordinates of every vertex, and circle the vertex maximizing profit. What is the maximum profit possible?

6. Considere a uma empresa que produz tapetes. As estimativas de demanda para os 12 meses do calendário são $d_1, ..., d_{12}$. A empresa tem 20 empregados, sendo que cada um produz 20 tapetes por mês a ganha um salário mensal de $2000. A empresa não tem estoque inicial de tapetes no começo do ano. Adicionalmente, a empresa lida com a variação de demanda da seguinte maneira:

   - Hora extra: Neste regime, o salário é 80% mais alto do que o regular (pago proporcional por tapete extra) e os trabalhadores podem trabalhar no máximo 30% do seu tempo de trabalho fazendo hora extra.
   - Contratação e demissão: os custos trabalhistas são $320 e $400, respectivamente, por trabalhador.
   - Estocagem de produtos: custo de $8 por tapete por mês.

Apresente uma formulação para que a empresa minimize seu custo de operação em função das demandas mensais.

Adaptado de Dasgupta et al. (2006).

7. (Dasgupta et al., 2006, ex. 7.6)

   Give an example of a linear program in two variables whose feasible region is infinite, but such that there is an optimum solution of bounded cost.

8. (Dasgupta et al., 2006, ex. 7.7)

   Find necessary and sufficient conditions on the reals a and b under which the linear program

   max $x + y$

   s.t.

   $ax + by \leq 1$

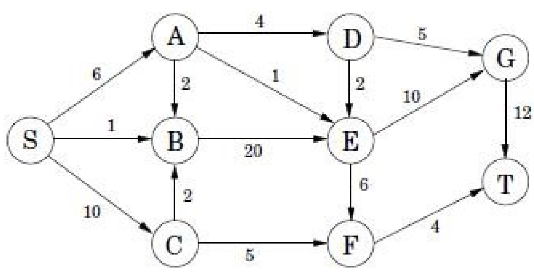
   $x, y \geq 0$

   (a) Is infeasible.

   (b) Is unbounded.

   (c) Has a finite and unique optimal solution.

9. A figura abaixo mostra um grafo direcionado que representa uma rede de oleodutos. Não há lugar para se armazenar petróleo durante a rota nos vértices do grafo, de maneira que a soma do volume que entra em cada vértice (exceto nos casos de S e T) deve ser igual à soma do que sai dele.



O peso de cada aresta indica a capacidade (vazão máxima por minuto) de cada trecho do oleoduto.

Queremos determinar a maior quantidade de petróleo de que é possível enviar de S para T em um minuto.

Modele o problema por meio de um Programa Linear.

3 PL: Conceitos Fundamentais e Introdução ao Simplex

10. Formule o PL do Exercício 2 na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

11. Formule o PL do Exercício 3 na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

12. Formule o PL do Exercício 4 na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

13. Formule o PL do Exercício 5 na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

14. Formule o PL do Exercício 6 na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

15. Formule o PL do Exercício 9 na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

16. Formule o PL em (Matoušek and Gärtner, 2007, Seção 2.1) na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

17. Formule o PL em (Matoušek and Gärtner, 2007, Seção 2.2) na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

18. Formule o PL em (Matoušek and Gärtner, 2007, Seção 2.3) na forma equacional. Explicite os valores de A, b e c, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Cap. 4).

19. Considere o exemplo de aproximação de um conjunto de pontos no plano por uma reta discutido em (Matoušek and Gärtner, 2007, Seção 2.4). Se o número de pontos no plano é n, quantas equações e quantas variáveis terá o PL correspondente na forma equacional?

20. Resolva o PL do Exercício 2 (formulado no Exercício 10). Em cada iteração, indique explicitamente os valores de B, x_B, p, Q, x_N, z, z_0, r, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Seção 5.5).

21. Resolva o PL do Exercício 3 (formulado no Exercício 11). Em cada iteração, indique explicitamente os valores de B, x_B, p, Q, x_N, z, z_0, r, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Seção 5.5).

22. Resolva o PL do Exercício 4 (formulado no Exercício 12). Em cada iteração, indique explicitamente os valores de B, x_B, p, Q, x_N, z, z_0, r, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Seção 5.5).

23. Resolva o PL do Exercício 5 (formulado no Exercício 13). Em cada iteração, indique explicitamente os valores de B, x_B, p, Q, x_N, z, z_0, r, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Seção 5.5).

24. Resolva o PL do Exercício 6 (formulado no Exercício 14). Em cada iteração, indique explicitamente os valores de B, x_B, p, Q, x_N, z, z_0, r, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Seção 5.5).

25. Resolva o PL formulado no Exercício 17. Em cada iteração, indique explicitamente os valores de B, x_B, p, Q, x_N, z, z_0, r, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Seção 5.5).

4 PL: Detalhes do Simplex

26. Resolva o PL abaixo usando o simplex e diga se tem solução única, se tem infinitas soluções, se é ilimitado, se é inviável e se tem algum vértice degenerado. Indique como o simplex pode ser usado para decidir isso.

$$\begin{aligned}
&\max\ 3x - y\\
&\text{s.t.}\\
&x - y \leq 1\\
&2x - y \leq 3\\
&x, y \geq 0
\end{aligned}$$

27. Resolva o PL abaixo usando o simplex e diga se tem solução única, se tem infinitas soluções, se é ilimitado, se é inviável e se tem algum vértice degenerado. Indique como o simplex pode ser usado para decidir isso.

$$\begin{aligned}
&\max\ x + 2y\\
&\text{s.t.}\\
&-2x + y \leq 0\\
&2x - 5y \leq 0\\
&x \leq 5\\
&y \leq 4\\
&x, y \geq 0
\end{aligned}$$

28. Resolva o PL abaixo usando o simplex e diga se tem solução única, se tem infinitas soluções, se é ilimitado, se é inviável e se tem algum vértice

degenerado. Indique como o simplex pode ser usado para decidir isso.

$$\max \; -x - 2y - 3z$$
s.t.
$$x + y + z \geq 2$$
$$x, y, z \geq 0$$

29. Resolva o PL formulado no Exercício 16. Em cada iteração, indique explicitamente os valores de B, x_B, p, Q, x_N, z, z_0, r, de acordo com a notação utilizada em (Matoušek and Gärtner, 2007, Seção 5.5).

5 PL: Dualidade e PLI

30. (Dasgupta et al., 2006, ex. 7.11)

    Write the dual to the following linear program.

$$\max \; x + y$$
s.t.
$$2x + y \leq 3$$
$$x + 3y \leq 5$$
$$x, y \geq 0$$

    Find the optimal solutions to both primal and dual LPs.

31. (Dasgupta et al., 2006, ex. 7.12)

    For the linear program

$$\max \; x_1 - 2x_3$$
s.t.
$$x_1 - x_2 \leq 1$$
$$2x_2 - x_3 \leq 1$$
$$x_1, x_2, x_3 \geq 0$$

    prove that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal.

32. Qual o dual do problema do conjunto independente máximo?

33. Qual o dual do problema do caminho mínimo?

34. (Dasgupta et al., 2006, ex. 7.23)

    A vertex cover of an undirected graph $G = (V, E)$ is a subset of the vertices which touches every edge — that is, a subset $S \subset V$ such that for each edge $\{u, v\} \in E$, one or both of u, v are in S. Show that the problem of finding the minimum vertex cover in a bipartite graph reduces to maximum flow. (Hint: Can you relate this problem to the minimum cut in an appropriate network?)

35. (Dasgupta et al., 2006, ex. 7.29)

    Hollywood. A film producer is seeking actors and investors for his new movie. There are n available actors; actor i charges s_i dollars. For funding, there are m available investors. Investor j will provide p_j dollars, but only on the condition that certain actors $L_j \subseteq \{1, 2, \ldots, n\}$ are included in the cast (all of these actors L_j must be chosen in order to receive funding from investor j). The producer's profit is the sum of the payments from investors minus the payments to actors. The goal is to maximize this profit.

    (a) Express this problem as an integer linear program in which the variables take on values $\{0, 1\}$.

    (b) Now relax this to a linear program, and show that there must in fact be an integral optimal solution (as is the case, for example, with maximum flow and bipartite matching).

36. No caso geral problema de emparelhamento de peso máximo discutido em (Matoušek and Gärtner, 2007, Sec. 3.2), onde o grafo tem n vértices (n par com $n/2$ vértices em cada parte) e m arestas, quantas variáveis e quantas restrições terá o correspondente programa linear em sua forma equacional?

37. Considere a versão com pesos (isto é, cada vértice tem um peso) do problema da cobertura mínima por vértices discutido em (Matoušek and Gärtner, 2007, Sec. 3.3). Prove que

$$w(S_{\text{LP}}) \leq 2w(S_{\text{OPT}})$$

onde S_{OPT} é a solução ótima do problema linear inteiro, S_{LP} é a solução ótima do respectivo programa linear relaxado e $w(\cdot)$ é a função objetivo (de ambos).

6 Enumeração / Backtracking

38. (Kreher and Stinson, 1999, ex. 2.1)

    Let $S = \{2, 3, 5, 7, 11, 13\}$. Determine the rank of the subset $\{3, 7, 13\}$ among the subsets of S in lexicographic order, and verify that unrank(rank($\{3, 7, 13\}$)) $= \{3, 7, 13\}$.

39. (Kreher and Stinson, 1999, ex. 2.2)

    Find all possible Gray codes for $n = 4$.

40. (Kreher and Stinson, 1999, ex. 2.4)

    Find the successor and the rank of the binary vector 01010110 in the Gray code G^8.

41. (Kreher and Stinson, 1999, ex. 2.6)

    Find a successor algorithm for the co-lex ordering of k-subsets of an n-element set.

42. (Kreher and Stinson, 1999, ex. 2.7)

    What is the rank of $\{3, 6, 7, 9\}$ considered as a 4-subset of $\{0, ..., 12\}$, in lexicographic, co-lex and revolving door order? What is its successor in each of these orders?

43. (Kreher and Stinson, 1999, ex. 2.10)

    Another way to order the subsets of an n-element set is to order them first in increasing size, and then in lexicographic order for each fixed size. For example, when $n = 3$, this ordering for the subsets of $S = \{1, 2, 3\}$ is:

    $$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

Develop unranking, ranking and successor algorithms for the subsets with respect to this ordering.

44. (Kreher and Stinson, 1999, ex. 2.11)

    Find the rank and successor of the permutation $[2, 4, 6, 7, 5, 3, 1]$ in lexicographic and Trotter-Johnson order.

45. (Kreher and Stinson, 1999, ex. 2.12)

    A derangement is a permutation $[\pi[1], \pi[2], ..., \pi[n]]$ of the set $\{1, 2, 3, \ldots, n\}$ such that $\pi[i] \neq i$, for all $t = 1, 2, ..., n$. Let D_n denote the number of derangements of an n-element set. Prove the recurrence relation $D_n = (n-1)(D_{n-1} + D_{n-2})$. Then, use this recurrence relation to develop an algorithm to generate all the derangements.

46. (Kreher and Stinson, 1999, ex. 2.13)

    A *multiset* is a set with (possibly) repeated elements. A k-multiset is one that contains k elements (counting repetitions). Thus, for example, $\{1, 2, 3, 1, 1, 3\}$ is a 6-multiset. The k-multisets of an n-set can be ordered lexicographically, by sorting the elements in each multiset in non-decreasing order and storing the result as a list of length k. Develop unranking, ranking and successor algorithms for the k-multisets of an n-set.

47. (Kreher and Stinson, 1999, ex. 2.14)

    k-permutations were defined in Section 1.2.1. Assuming that $k < n$, develop a minimal change algorithm to generate the k-permutations of an n-set with a minimal change algorithm. At each step, this algorithm should change exactly one element.

48. (Kreher and Stinson, 1999, ex. 4.1)

    Define choice sets and describe backtracking algorithms for the following problems:

    (a) Find all ways of placing n mutually non-attacking queens on an n by n chess board.

(b) Find all self-avoiding walks of length n. (A self-avoiding walk is described by a sequence of edges in the Euclidean plane, beginning at the origin, such that each of the edges is a vertical or horizontal line segment of length one, and such that no point in the plane is visited more than once. There are precisely three such walks of length one, 12 walks of length two, and 36 walks of length three.)

(c) Find all k-vertex colorings of a graph G.

49. (Kreher and Stinson, 1999, ex. 4.3)

Determine the complexity of Algorithm 4.8, with and without the assumption that the objects are sorted according to their profit / weight ratios.

7 Branch-and-bound

50. (Kreher and Stinson, 1999, ex. 4.4)

Use Algorithm 4.9 to solve the following instances of the Knapsack (optimization) problem.

(a)

Profits	122	2	144	133	52	172	169	50	11	87	127	31	10	132	59
Weights	63	1	71	73	24	79	82	23	6	43	66	17	5	65	29
Capacity	323														

(b)

Profits	143	440	120	146	266	574	386	512	106	418	376	124	48	535	55
Weights	72	202	56	73	144	277	182	240	54	192	183	67	23	244	29
Capacity	1019														

(c)

Profits	818	460	267	75	621	280	555	214	721	427	78	754	704	44	371
Weights	380	213	138	35	321	138	280	118	361	223	37	389	387	23	191
Capacity	1617														

51. (Kreher and Stinson, 1999, ex. 4.11)

An edge-decomposition of the complete graph K_n into triangles is called a *Steiner triple system* of order n (or, $STS(n)$). More formally, an $STS(n)$ is a pair (P, B) in which P is an n-element set of points; B is a collection of $n(n-1)/6$ 3-element subsets of P called *triples* (or *blocks*); and every pair of points is contained in exactly one triple.

(a) Write a backtracking algorithm to find all $STS(n)$ (on a given set of n vertices), and use your algorithm to determine the number of different $STS(7)$.

(b) Define a graph $G = (V, E)$, where V consists of the $\binom{n}{3}$ 3-subsets of an n-set, and two vertices are adjacent if and only if the intersection of the corresponding subsets has cardinality at most one. Show that an $STS(n)$ is equivalent to a (maximum) clique in G having size $n(n-1)/6$.

(c) Using any of the clique-finding algorithms described in this chapter, determine the number of different $STS(7)$.

52. (Kreher and Stinson, 1999, ex. 4.15)

The Minimum Spanning Tree problem consists of a complete graph K_n with a cost function defined on its edges. The problem is to find a set of $n - 1$ edges that form a tree (i.e., which do not contain a circuit) such that the sum of their costs is minimized. It is well known that this problem can be solved by a greedy algorithm, which considers the edges in increasing order of cost, adding each edge to the tree being constructed if and only if it does not create a circuit.

Suppose that $[x_0, \ldots, x_{\ell-1}]$ is a partial solution for the Traveling Salesman problem. Describe a bounding function based on the idea of computing the mini- minimum spanning tree in the subgraph induced by the vertices in the set

$$\{0, \ldots, n-1\} \setminus \{x_1, \ldots, x_{\ell-2}\}.$$

8 Gulosos / Programação Dinâmica

53. (Kreher and Stinson, 1999, ex. 1.15)

Use a dynamic programming algorithm to solve the following instance of the Knapsack (optimal value) problem:

Profits	1, 2, 3, 5, 7, 10;
Weights	2, 3, 5, 8, 13, 16;
Capacity	30

Then, using the table of values $P[j, m]$, solve the Knapsack (optimization) problem for the same problem instance.

Nota: Veja (Kreher and Stinson, 1999, Seção 1.3).

54. (Cormen, Leiserson, Rivest, and Stein, 2009, ex. 15.1-2)

    Show, by means of a counterexample, that the following "greedy" strategy does not always determine an optimal way to cut rods. Define the density of a rod of length i to be p_i/i, that is, its value per inch. The greedy strategy for a rod of length n cuts off a first piece of length i, where $1 \leq i \leq n$, having maximum density. It then continues by applying the greedy strategy to the remaining piece of length $n - i$.

55. (Cormen et al., 2009, ex. 15.1-3)

    Consider a modification of the rod-cutting problem in which, in addition to a price p_i for each rod, each cut incurs a fixed cost of c. The revenue associated with a solution is now the sum of the prices of the pieces minus the costs of making the cuts. Give a dynamic-programming algorithm to solve this modified problem.

56. Baseado em: (Cormen et al., 2009, ex. 15.3-6)

    Imagine that you wish to exchange one currency for another. You realize that instead of directly exchanging one currency for another, you might be better off making a series of trades through other currencies, winding up with the currency you want. Suppose that you can trade n different currencies, numbered $1, 2, \ldots, n$, where you start with currency 1 and wish to wind up with currency n. You are given, for each pair of currencies i and j, an exchange rate r_{ij}, meaning that if you start with d units of currency i, you can trade for dr_{ij} units of currency j.

    Resolva usando programação dinâmica.

57. (Cormen et al., 2009, ex. 16.1-2)

    [Para o problema "activity-selection problem" da Seção 16.1:]

    Suppose that instead of always selecting the first activity to finish, we instead select the last activity to start that is compatible with all previously selected activities. Describe how this approach is a greedy algorithm, and prove that it yields an optimal solution.

58. (Cormen et al., 2009, ex. 16.1-1)

Prove that the fractional knapsack problem has the greedy-choice property.

Referências

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