Structure and algorithms for (cap, even hole)-free graphs

Kathie Cameron* Murilo V. G. da Silva† Shenwei Huang‡
Kristina Vušković§

July 09, 2017

Abstract

A graph is even-hole-free if it has no induced even cycles of length 4 or more. A cap is a cycle of length at least 5 with exactly one chord and that chord creates a triangle with the cycle. In this paper, we consider (cap, even hole)-free graphs, and more generally, (cap, 4-hole)-free odd-signable graphs. We give an explicit construction of these graphs. We prove that every such graph G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$, and hence $\chi(G) \leq \frac{3}{2}\omega(G)$, where $\omega(G)$ denotes the size of a largest clique in G and $\chi(G)$ denotes the chromatic number of G. We give an $O(nm)$ algorithm for q-coloring these graphs for fixed q and an $O(nm)$ algorithm for maximum weight stable set, where n is the number of vertices and m is the number of edges of the input graph. We also give a polynomial-time algorithm for minimum coloring.

Our algorithms are based on our results that triangle-free odd-signable graphs have treewidth at most 5 and thus have clique-width at most 48, and that (cap, 4-hole)-free odd-signable graphs G without clique cutsets have treewidth at most $6\omega(G) - 1$ and clique-width at most 48.

Keywords: even-hole-free graph, structure theorem, decomposition, combinatorial optimization, coloring, maximum weight stable set, treewidth, clique-width

1 Introduction

In this paper all graphs are finite and simple. We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G. A graph G is F*-free* if it does not contain F, and for a family of graphs $\mathcal{F}$, G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. A *hole* is a chordless cycle of length at least four. A hole is *even* (respectively, *odd*) if it has an even (respectively, odd)

*Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada, N2L 3C5. Research supported by the Natural Sciences and Engineering Research Council of Canada (grant RGPIN-2016-06517).

†Department of Computer Science, Federal University of Technology-Paraná, Curitiba, Brazil.

‡School of Computer Science and Engineering, University of New South Wales, Sydney, NSW 2052, Australia.

§School of Computing, University of Leeds, Leeds LS2 9JT, UK; and Faculty of Computer Science (RAF), Union University, Knez Mihajlova 6/VI, 11000 Belgrade, Serbia. Partially supported by EPSRC grant EP/N019660/1 and Serbian Ministry of Education, Science and Technological Development projects 174033 and III44006.

number of vertices. A *cap* is a graph that consists of a hole H and a vertex x that has exactly two neighbors in H, that are furthermore adjacent. The graph C_n is a hole of length n, and is also called an *n-hole*. In this paper we study the class of (cap, even hole)-free graphs, and more generally the class of (cap, 4-hole)-free odd-signable graphs, which we define later.

Let G be a graph. We use n to denote the number of vertices of G and m the number of edges of G. A set $S \subseteq V(G)$ is a *clique* of G if all pairs of vertices of S are adjacent. The size of a largest clique in a graph G is denoted by $\omega(G)$, and is sometimes called the *clique number* of G. We say that G is a *complete graph* if $V(G)$ is a clique. We denote by K_n the complete graph on n vertices. The graph K_3 is also called a *triangle*. A set $S \subseteq V(G)$ is a *stable set* of G if no two vertices of S are adjacent. The size of a largest stable set of G is denoted by $\alpha(G)$. A *q-coloring* of G is a function $c : V(G) \longrightarrow \{1, \dots, q\}$, such that $c(u) \neq c(v)$ for every edge uv of G. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number q for which there exists a q-coloring of G.

The class of (cap, odd hole)-free graphs has been studied extensively in literature. This is precisely the class of Meyniel graphs, where a graph G is *Meyniel* if every odd length cycle of G, that is not a triangle, has at least two chords. These graphs were proven to be perfect by Meyniel [30] and Markosyan and Karapetyan [28]. Burlet and Fonlupt [5] obtained the first polynomial-time recognition algorithm for Meyniel graphs, by decomposing these graphs with amalgams (that they introduced in the same paper). Subsequently, Roussel and Rusu [33] obtained a faster algorithm for recognizing Meyniel graphs (of complexity $O(m^2)$), that is not decomposition-based. Hertz [23] gave an $O(nm)$ algorithm for coloring and obtaining a largest clique of a Meyniel graph. This algorithm is based on contractions of even pairs. It is an improvement on the $O(n^8)$ algorithm of Hoàng [24]. Roussel and Rusu [34] gave an $O(n^2)$ algorithm that colors a Meyniel graph without using even pairs. This algorithm "simulates" even pair contractions and it is based on lexicographic breadth-first search and greedy sequential coloring.

Algorithms have also been given which find a minimum coloring of a Meyniel graph, but do not require that the input graph be known to be Meyniel. A *Meyniel obstruction* is an induced subgraph which is an odd cycle with at most one chord. A *strong stable set* in a graph G is a stable set which intersects every (inclusion-wise) maximal clique of G. Cameron and Edmonds [6] gave an $O(n^2)$ algorithm which for any graph, finds either a strong stable set or a Meyniel obstruction. This algorithm can be applied at most n times to find, in any graph, either a clique and coloring of the same size or a Meyniel obstruction. Cameron, Lévêque and Maffray [7] showed that a variant of the Roussel-Rusu coloring algorithm for Meyniel graphs [34] can be enhanced to find, for any input graph, either a clique and coloring of the same size or a Meyniel obstruction. The worst-case complexity of the algorithm is still $O(n^2)$.

In [11], Conforti, Cornuéjols, Kapoor and Vušković generalize Burlet and Fonlupt's decomposition theorem for Meyniel graphs [5] to the decomposition by amalgams of all cap-free graphs. This theorem is the basis for polynomial-time recognition algorithms for cap-free odd-signable graphs and (cap, even hole)-free graphs. Since triangle-free graphs are cap-free, it follows that the problems of coloring and of finding the size of a largest stable set are both NP-hard for cap-free graphs. In [14], Conforti, Gerards and Pashkovich show how to obtain a polynomial-time algorithm for solving the maximum weight stable set problem on any class of graphs that is decomposable by amalgams into basic graphs for which one can solve the maximum weight stable set problem in polynomial time. This leads to the first known non-polyhedral algorithm for the maximum weight stable set problem for Meyniel graphs. Furthermore, using the decomposition theorems from [11] and [12], they obtain a polynomial-time algorithm for solving the maximum weight stable set problem for (cap, even hole)-free graphs (and more generally cap-free odd-signable graphs). For a survey on even-hole-free graphs and odd-signable graphs, see [39].

Aboulker, Charbit, Trotignon and Vušković [1] gave an $O(nm)$-time algorithm whose input is a weighted graph G and whose output is a maximum weighted clique of G or a certificate proving that G is not 4-hole-free odd-signable. (The crux of this algorithm was actually developed by da Silva and Vušković in [20].)

In Section 3, we give an explicit construction of (cap, 4-hole)-free odd-signable graphs, based on [11] and [12]. From this, in Section 4, we derive that every such graph G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$, and hence $\chi(G) \leq \frac{3}{2}\omega(G)$. It follows that G can be colored with at most $\frac{3}{2}\omega(G)$ colors using the greedy coloring algorithm.

In Section 5, we prove that triangle-free odd-signable graphs have treewidth at most 5 and thus have clique-width at most 48 [15]. We also prove that (cap, 4-hole)-free odd-signable graphs G without clique cutsets have clique-width at most 48 and treewidth at most $6\omega(G) - 1$.

In Section 6, we give an $O(nm)$ algorithm for q-coloring (cap, 4-hole)-free odd-signable graphs. We give a (first known) polynomial-time algorithm for finding a minimum coloring of these graphs (chromatic number). We also obtain an $O(nm)$ algorithm for the maximum weight stable set problem for (cap, 4-hole)-free odd-signable graphs. We observe that the algorithm in [14] proceeds by first decomposing the graph by amalgams, a step that takes $O(n^4m)$ time ($O(n^2m)$ to find an amalgam [16], which is called $O(n^2)$ times) and creates $O(n^2)$ indecomposable graphs. For each indecomposable graph, $O(n)$ maximum weight stable set problems must be solved, each of which can be done in $O(n + m)$ time. Thus the overall complexity of their algorithm is $O(n^4m)$. Finally, we observe that all our algorithms are robust in the sense that we do not need to assume that the input graph is (cap, 4-hole)-free odd-signable.

It is known that planar even-hole-free graphs have treewidth at most 49 [35]. We observe that (cap, even hole)-free graphs are not necessarily planar. It is not hard to check that the graph in Figure 1 is (triangle, even hole)-free and has a K_5-minor.

The complexity of the stable set problem and of the coloring problem remain open for even-hole-free graphs.

2 Odd-signable graphs

We *sign* a graph by assigning $0, 1$ weights to its edges. A graph is *odd-signable* if there exists a signing that makes the sum of the weights in every chordless cycle (including triangles) odd. Even-hole-free graphs are clearly odd-signable: assign weight 1 to each edge. To characterize odd-signable graphs in terms of excluded induced subgraphs, we now introduce two types of *3-path configurations* ($3PC$'s) and even wheels.

Let x and y be two distinct vertices of G. A $3PC(x, y)$ is a graph induced by three chordless xy-paths, such that any two of them induce a hole. We say that a graph G contains a $3PC(\cdot, \cdot)$ if it contains a $3PC(x, y)$ for some $x, y \in V(G)$. The $3PC(\cdot, \cdot)$'s are also known

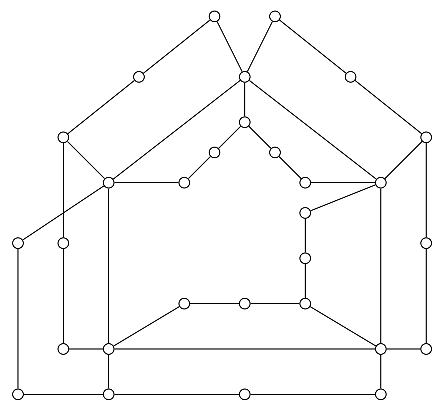

Figure 1: A (triangle even hole)-free graph that has a K_5-minor.

as *thetas*.

Let $x_1, x_2, x_3, y_1, y_2, y_3$ be six distinct vertices of G such that $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ induce triangles. A $3PC(x_1x_2x_3, y_1y_2y_3)$ is a graph induced by three chordless paths $P_1 = x_1, \ldots, y_1$, $P_2 = x_2, \ldots, y_2$ and $P_3 = x_3, \ldots, y_3$, such that any two of them induce a hole. We say that a graph G contains a $3PC(\Delta, \Delta)$ if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. The $3PC(\Delta, \Delta)$'s are also known as *prisms*.

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a vertex $x \notin V(H)$ having at least three neighbors in H, say $x_1, \ldots, x_r$. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate vertex x_l, $1 \leq l \leq r$. A wheel (H, x) is *even* if it has an even number of sectors.

It is easy to see that even wheels, thetas and prisms cannot be contained in even-hole-free graphs. In fact they cannot be contained in odd-signable graphs. The following characterization of odd-signable graphs states that the converse also holds, and it is an easy consequence of a theorem of Truemper [38].

Theorem 2.1 ([11]) *A graph is odd-signable if and only if it does not contain an even wheel, a theta or a prism.*

3 Construction of (cap, 4-hole)-free odd-signable graphs

Let G be a graph and $S \subseteq V(G)$. The subgraph of G induced by S is denoted by $G[S]$, and $G \setminus S = G[V(G) \setminus S]$. We say that S is a *vertex cutset* of G if $G \setminus S$ is disconnected. A *clique cutset* of G is a vertex cutset that is a clique of G. Note that an empty set is a clique, and hence every disconnected graph has a clique cutset. An graph with no clique cutset is called an *atom*.

Let $G = (V, E)$ be a graph and $K \subseteq V$ a clique cutset such that $G \setminus K$ is a disjoint union of two subgraphs H_1 and H_2 of G. We let G_i be the subgraph of G induced by $V(H_i) \cup K$ for $i = 1, 2$. We say that G is *decomposed into* G_1 and G_2 *via* K, and call this a *decomposition step*. We then recursively decompose G_1 and G_2 via clique cutsets until no clique cutset exists. This procedure can be represented by a rooted binary tree T_G where G is the root and the leaves are induced subgraphs of G that do not contain clique cutsets (that is, atoms of G). Tarjan [37] showed that for any graph G, T_G can be found in $O(nm)$ time. Moreover, in each decomposition step Tarjan's algorithm produces an atom, and consequently T_G has at most $n-1$ leaves (or equivalently atoms).

Let A and B be disjoint subsets of vertices of a graph G. We say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and A is *anticomplete* to B if no vertex of A is adjacent to a vertex of B. A vertex which is adjacent to all other vertices is called *universal*.

A connected graph G has an *amalgam* (V_1, V_2, A_1, A_2, K) if the following hold:

- $V(G) = V_1 \cup V_2 \cup K$, where V_1, V_2 and K are disjoint sets, and $|V_1| \geq 2$, $|V_2| \geq 2$.
- K is a (possibly empty) clique of G.
- For $i = 1, 2$, $\emptyset \neq A_i \subseteq V_i$.
- A_1 is complete to A_2, and these are the only edges with one end in V_1 and the other in V_2.
- K is complete to $A_1 \cup A_2$ (note that vertices of K may have other neighbors in $V_1 \cup V_2$).

A graph is *chordal* if it is hole-free. A graph is *2-connected* if it has at least 3 vertices and remains connected whenever fewer than 2 vertices are removed. A *basic cap-free graph* G is either a chordal graph or a 2-connected triangle-free graph together with at most one additional vertex, that is adjacent to all other vertices of G.

Theorem 3.1 ([11]) *A connected cap-free graph that is not basic has an amalgam.*

A *module* (or *homogeneous set*) in a graph G is a set $M \subseteq V(G)$, such that $2 \leq |M| \leq |V(G)| - 1$ and every vertex of $V(G) \setminus M$ is either adjacent to all of M or none of M. Note that a module with $2 \leq |M| \leq |V(G)| - 2$ (i.e. a *proper module*) is a special case of an amalgam. A *clique module* is a module that induces a clique.

Let G be a 4-hole-free graph and (V_1, V_2, A_1, A_2, K) an amalgam of G. Without loss of generality we may assume that A_1 induces a clique, and hence either $A_1 \cup K$ is a clique cutset or A_1 is a proper clique module. So, Theorem 3.1 particularized to 4-hole-free graphs, and the well-known fact that every chordal graph that is not a clique has a clique cutset, gives the following decomposition.

Theorem 3.2 *If G is (cap, 4-hole)-free graph, then either G has a clique cutset or a proper clique module, or G is a complete graph or a (triangle, 4-hole)-free graph together with at most one additional vertex that is adjacent to all other vertices of G.*

We now give a complete structural description of (cap, 4-hole)-free graphs that do not have a clique cutset, by considering the proof of Theorem 3.1 from [11] particularized to 4-hole-free graphs.

An *expanded hole* consists of nonempty disjoint sets of vertices $S_1, \ldots, S_k$, $k \geq 4$, not all singletons, such that for all $1 \leq i \leq k$, the graphs $G[S_i]$ are connected and, for $i \neq j$, S_i is complete to S_j if $j = i+1$ or $j = i-1$ (modulo k), and anticomplete otherwise. We also say that $G[S]$ is an expanded hole. Note that if an expanded hole is 4-hole-free, then S_i is a clique for every $i = 1, \ldots, k$. The following statement can be extracted from the proofs of Lemma 5.1 and Theorem 7.1 in [11].

Lemma 3.3 *Let G be a (cap, 4-hole)-free graph. Suppose that $S = \cup_{i=1}^{k} S_i$ is an inclusion-wise maximal expanded hole of G such that $|S_2| \geq 2$. Let U be the set of vertices of G that are complete to S. Then G has an amalgam (V_1, V_2, A_1, A_2, K) where $S_2 = A_2$ and $K \subseteq U$. In particular, either $K \cup S_2$ is a clique cutset or S_2 is a proper clique module.*

Let M be a proper clique module of a graph G. The block of decomposition of G with respect to M is the graph $G' = G \setminus (M \setminus \{u\})$, where u is any vertex of M. For a graph G and a vertex u of G, we denote $N_G(u)$ (or $N(u)$ when clear from context) by the set of neighbors of u in G. Also $N[u] = N(u) \cup \{u\}$. The degree $d_G(u)$ of u is $|N_G(u)|$.

Lemma 3.4 *Let M be a proper clique module of a graph G, and let G' be the block of decomposition with respect to this module. If G does not have a clique cutset, then G' does not have a clique cutset.*

Proof: Suppose K is a clique cutset of G'. Let u be the vertex of M that is in G'. If $u \notin K$ then $N_{G'}(u) \setminus K$ is in the same connected component of $G' \setminus K$ as u, and hence K is a clique cutset of G. If $u \in K$ then $M \cup K$ is a clique cutset of G. $\square$

Theorem 3.5 *Let G be a (cap, 4-hole)-free graph that contains a hole. Let F be a maximal vertex subset of $V(G)$ that induces a 2-connected triangle-free graph, U the set of vertices of $V(G) \setminus F$ that are complete to F, D the set of vertices of $V(G) \setminus F$ that have at least two neighbors in F but are not complete to F, and $S = V(G) \setminus (F \cup U \cup D)$. Then the following hold:*

(i) U is a clique.

(ii) U is complete to $D \cup F$.

(iii) If G does not have a clique cutset, then for every $d \in D$, there is a vertex $u \in F$ and $D' \subseteq D$ that contains d such that $D' \cup \{u\}$ is a clique module of G. In particular, for every $d' \in D'$, $N[d'] = N[u]$.

(iv) If G does not have a clique cutset, then $S = \emptyset$.

(v) If G does not have a clique cutset, F does not have a clique cutset.

Proof: Since G is 4-hole-free and F contains nonadjacent vertices, clearly U must be a clique, and hence (i) holds.

Claim: *For every $d \in D$, $G[F]$ contains a hole H such that $G[V(H) \cup \{d\}]$ is an expanded hole of G.*
Proof of the Claim: Let d be any vertex of D, and assume that there is no hole H contained in $G[F]$ such that $G[V(H) \cup \{d\}]$ is an expanded hole. Note that $G[F \cup \{d\}]$ contains a triangle d, x, y, for otherwise the maximality of F is contradicted. Since $G[F]$ is 2-connected

and triangle-free, x and y are contained in a hole H of $G[F]$. It is easy to see that since G is cap-free and we are assuming that $G[V(H) \cup \{d\}]$ is not an expanded hole, it follows that d is complete to $V(H)$. Let F' be a maximal subset of F such that $G[F']$ contains H, is 2-connected and d is complete to F'. Since $F \neq F'$ and both $G[F]$ and $G[F']$ are 2-connected, some $z \in F \setminus F'$ belongs to a hole H' that contains an edge of $G[F']$. As before, it follows that d is complete to $V(H')$, and hence $F' \cup V(H')$ contradicts the choice of F'. This completes the proof of the Claim.

By the Claim, every vertex d of D has two nonadjacent neighbors in F, and since G is 4-hole-free, it follows that every vertex of U is adjacent to d. Therefore, (ii) holds.

Now suppose that G does not contain a clique cutset. By the Claim and Lemma 3.3, (iii) holds. Let $D' \cup \{u\}$ be a proper clique module from (iii). Then the block of decomposition with respect to this module is the graph $G \setminus D'$. So by performing a sequence of clique module decompositions, we get the graph $G' = G \setminus D$. By Lemma 3.4, G' does not have a clique cutset. Suppose that $S \neq \emptyset$. Note that every vertex in S has at most one neighbor in F. Let C be a connected component of $G[S]$. By the maximality of F, there is at most one vertex in F, say y, that has a neighbor in C. So $U \cup \{y\}$ is a clique cutset of G', a contradiction. If no component of $G[S]$ is adjacent to a vertex of F, then U is a clique cutset of G', a contradiction. Therefore, (iv) holds. Finally, suppose that F has a clique cutset K. Then by (i) and (iv), $K \cup U$ is a clique cutset of G', a contradiction. Therefore, (v) holds. □

We say that the graph G' is obtained from a graph G by *blowing up vertices of G into cliques* if G' consists of the disjoint union of cliques K_u, for every $u \in V(G)$, and all edges between cliques K_u and K_v if and only if $uv \in E(G)$. This is also referred to as substituting clique K_u for vertex u (for all u). The graph G' is obtained from a graph G by *adding a universal clique* if G' consists of G together with (a possibly empty) clique K, and all edges between vertices of K and vertices of G. Note that both of these operations preserve being (cap, 4-hole)-free, i.e., G is (cap, 4-hole)-free if and only if G' is (cap, 4-hole)-free.

Theorem 3.6 *Let G be a (cap, 4-hole)-free graph that contains a hole and has no clique cutset. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. Then G is obtained from F by first blowing up vertices of F into cliques, and then adding a universal clique. Furthermore, any graph obtained by this sequence of operations starting from a (triangle, 4-hole)-free graph with at least 3 vertices and no clique cutset is (cap, 4-hole)-free and has no clique cutset.*

Proof: Let F' be a maximal 2-connected triangle-free induced subgraph of G that contains F. By Theorem 3.5 (v), F' does not have a clique cutset, and hence $F' = F$. So the first statement follows from Theorem 3.5. The second statement follows from an easy observation that blowing up vertices into cliques and adding a universal clique preserves being (cap, 4-hole)-free and having no clique cutset. □

Triangle-free odd-signable graphs were studied in [12] where the following construction was obtained. A chordless xz-path P is an *ear* of a hole H contained in a graph G if $V(P) \setminus \{x, z\} \subseteq V(G) \setminus V(H)$, vertices $x, z \in V(H)$ have a common neighbor y in H, and $(V(H) \setminus \{y\}) \cup V(P)$ induces a hole H' in G. The vertices x and z are the *attachments* of P in

H, and H' is said to be obtained by *augmenting* H with P. A graph G is said to be obtained from a graph G' by an *ear addition* if the vertices of $G \setminus G'$ are the intermediate vertices of an ear of some hole H in G', say an ear P with attachments x and z in H, and the graph G contains no edge connecting a vertex of $V(P) \setminus \{x, z\}$ to a vertex of $V(G') \setminus \{x, y, z\}$, where $y \in V(H)$ is adjacent to x and z. An ear addition is *good* if

- y has an odd number of neighbors in P,
- G' contains no wheel (H_1, v), where $x, y, z \in V(H_1)$ and v is adjacent to y, and
- G' contains no wheel (H_2, y), where x, z are neighbors of y in H_2.

The complete bipartite graph $K_{4,4}$ with a perfect matching removed is called the *cube*. Note that the cube contains 4-holes.

Theorem 3.7 (Theorem 6.4 in [12]) *Let G be a triangle-free graph with at least three vertices that is not the cube and has no clique cutset. Then, G is odd-signable if and only if it can be obtained, starting from a hole, by a sequence of good ear additions.*

4 Bound on the chromatic number

It is well-known that the class of (triangle, 4-hole)-free graphs has unbounded chromatic number [25]. In [29] it is shown that (triangle, even hole)-free graphs have a vertex of degree at most 2. We now show that (triangle, 4-hole)-free odd-signable graphs that contain at least one edge have an edge whose ends each have degree at most 2. This will imply that every (cap, 4-hole)-free odd-signable graph (and in particular every (cap, even hole)-free graph) G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$, and hence that every graph in this class has a proper coloring that uses at most $\frac{3}{2}\omega(G)$ colors.

Theorem 4.1 *Every (cap, 4-hole)-free odd-signable graph G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$.*

Proof: Given a graph G, let us say that a vertex v of G is *nice* if its degree is at most $\frac{3}{2}\omega(G) - 1$. We prove that *if G is a (cap, 4-hole)-free odd-signable graph, then either G is complete or it has at least two nonadjacent nice vertices.* Assume that this does not hold and let G be a minimum counterexample.

Suppose that G has a clique cutset K. Let $C_1, \dots, C_k$ be the connected components of $G \setminus K$, and for $i = 1, \dots, k$, let $G_i = G[C_i \cup K]$. Since G is a minimum counterexample, for every i, G_i is either a complete graph or it has at least two nonadjacent nice vertices. So there is a vertex $v_i \in C_i$ that is nice in G_i, and hence in G as well. But then G has at least two nonadjacent nice vertices, a contradiction. Therefore, G does not have a clique cutset.

This also implies that G cannot be chordal, since every chordal graph is either complete or has a clique cutset. So, G contains a hole. Let F be a maximal induced subgraph of G that is triangle-free and has no clique cutset. By Theorem 3.6, G is obtained from F by blowing up vertices of F into cliques and adding a universal clique U. Note that if a vertex u is nice in $G \setminus U$, then it is nice in G, and hence by the choice of G, $U = \emptyset$. For $u \in V(F)$, let K_u be the clique that the vertex u is blown up into.

Claim: *If u_1, u, v, v_1 is a path of F such that u and v are each of degree 2 in F, then u or v is nice in G.*

Proof of the Claim: Since $|K_u| + |K_v| \leq \omega(G)$, we may assume that $|K_u| \leq \frac{1}{2}\omega(G)$. But then $d_G(v) = |K_u| + |K_v| - 1 + |K_{v_1}| \leq \frac{3}{2}\omega(G) - 1$. This completes the proof of the Claim.

By Theorem 3.7 we consider the last ear P in the construction of F. Say that P is an ear of hole H and its attachments in H are x and z. Let y be the common neighbor of x and z in H. Since P is a good ear, y has an odd number of neighbors in P. Let $y_1, \ldots, y_k$ be the neighbors of y in P in the order when traversing P from x to z (so $y_1 = x$ and $y_k = z$). Since F is (triangle, 4-hole)-free, both the y_1y_2-subpath of P and the $y_{k-1}y_k$-subpath of P contain an edge whose ends are each of degree 2. It then follows by the Claim that G has at least two nonadjacent nice vertices, a contradiction. □

Corollary 4.2 *If G is a (cap, 4-hole)-free odd-signable graph, then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

Proof: It follows immediately from Theorem 4.1. □

Theorem 4.1 immediately gives a $\frac{3}{2}$-approximation algorithm for coloring (cap, 4-hole)-free odd-signable graphs. The algorithm greedily colors a particular ordering of vertices $v_1, v_2, \ldots, v_n$, where v_i is a vertex of minimum degree in $G[v_1, \ldots, v_i]$. It is clear that this ordering of vertices can be found in $O(n^2)$ time. Therefore, Theorem 4.1 ensures that the greedy algorithm properly colors the graph using at most $\frac{3}{2}\omega(G)$ colors in $O(n^2)$ time.

5 Treewidth and clique-width

A *triangulation* $T(G)$ of a graph G is obtained from G by adding edges until no holes remain. Clearly, $T(G)$ is a chordal graph on the same vertex set as G which contains all edges of G. The *treewidth* of a graph G is the minimum of $\omega(T(G)) - 1$ over all triangulations $T(G)$ of G. Treewidth k is equivalent to having a tree decomposition of width k, which is generally used in algorithms. Bodlaender [3] gave an algorithm which for fixed k, recognizes graphs of treewidth at most k, and constructs a width k tree decomposition; the algorithm is linear in $n = |V(G)|$ but exponential in k.

The *clique-width* of a graph G is the minimum number of labels required to construct G using the following four operations: creating a new vertex with label i, joining each vertex with label i to each vertex with label j, changing the label of every vertex labelled i to j, and taking the disjoint union of two labelled graphs. A sequence of these operations which constructs the graph using at most k labels is called a *k-expression* or *clique-width k-expression.*

Treewidth and clique-width have similar algorithmic implications. Problems that can be expressed in monadic second-order logic MSO_2 can be solved in linear time for any class of graphs with treewidth at most k [17]. Problems that can be expressed in the subset MSO_1 of MSO_2 which does not allow quantification over edge-sets can be solved in linear time for any class of graphs of clique-width at most k [18]. Kobler and Rotics [27] showed that certain other problems including chromatic number can be solved in polynomial time for any class of graphs of clique-width at most k [27]. All these algorithms are exponential in k, and those using clique-width require a k-expression as part of the input. This latter requirement was removed by Oum and Seymour [32] and then more efficiently by Oum [31], who gave an $O(n^3)$ algorithm which, for any input graph and fixed integer k, finds a clique-width 8^k-expression or states that the clique-width is greater than k.

Corneil and Rotics [15] improved a result of Courcelle and Olariu [19] to show that the clique-width of a graph G is at most $3 \times 2^{tw(G)-1}$. Further, they construct, in polynomial time, a k-expression of the stated size. Espelage, Gurski, and Wanke [21] gave an algorithm that takes as input a tree decomposition with width k, and gives a clique-width $2^{O(k)}$-expression [21] in linear time.

A chord of a cycle is called *short* if it creates a triangle with the cycle.

Theorem 5.1 *Every triangle-free odd-signable graph has treewidth at most 5.*

Proof: Let G^* be a triangle-free odd-signable graph. Apply the clique cutset decomposition algorithm to decompose G^* into atoms. We will show that each atom is contained in a chordal graph that has clique number at most 6. Gluing these chordal graphs together along the clique cutsets used to decompose G^* gives a chordal graph containing G^* with clique number at most 6, and this proves the theorem.

It is easy to check that the cube is contained in a chordal graph with clique number 4. So we may assume that an atom G has at least 3 vertices and is not the cube. Then by Theorem 3.7, G can be obtained from a hole H by a sequence of good ear additions. Let $P_1, \dots, P_q$ be the sequence of ears in the construction, with P_q being the last ear added. For each i, let H_i be the hole P_i is attached to, let x_i and z_i be the attachments of P_i in H_i, and let y_i be the common neighbor of x_i and z_i in H_i.

We now obtain a triangulation T of G with clique number at most 6. We construct T as follows. For each ear P_i, make x_i, y_i and z_i complete to $P_i \setminus \{x_i, z_i\}$, and add the edge x_iz_i. Call the edges x_iz_i type 1 edges. Choose any edge uv of H, and join u and v to all the vertices of $H \setminus \{u, v\}$; call these type 2 edges. For each i, let $S_i = \{x_i, y_i, z_i\}$, $G_i = G[V(H) \cup V(P_1) \cup \dots \cup V(P_i)]$ and $T_i = T[V(H) \cup V(P_1) \cup \dots \cup V(P_i)]$.

Claim 1: *For $1 \leq j \leq q$, S_j is a clique cutset in T that separates $H_j \setminus S_j$ from $P_j \setminus S_j$.*

Proof of Claim 1: To prove the claim we prove by induction on i the following statement: *for $1 \leq j \leq i \leq q$, S_j is a cutset in T_i that separates $H_j \setminus S_j$ from $P_j \setminus S_j$.*

By construction, the statement clearly holds for $i = 1$. Let $i > 1$, and inductively assume that for $1 \leq j \leq i-1$, S_j is a cutset in T_{i-1} that separates $H_j \setminus S_j$ from $P_j \setminus S_j$. Suppose that for some $j \leq i$, S_j is not a cutset in T_i that separates $H_j \setminus S_j$ from $P_j \setminus S_j$. Then clearly (by construction), $j \leq i - 1$. By the induction hypothesis, S_j is a cutset in T_{i-1} that separates $H_j \setminus S_j$ from $P_j \setminus S_j$. Let C_{H_j} (respectively, C_{P_j}) be the connected component of $T_{i-1} \setminus S_j$ that contains $H_j \setminus S_j$ (respectively, $P_j \setminus S_j$). Then, without loss of generality, $x_i \in C_{H_j}$ and $z_i \in C_{P_j}$. Since H_i is a hole of G_{i-1} that contains x_i and z_i, and S_j is a cutset in G_{i-1} which separates x_i and z_i, and y_j is adjacent to x_j and z_j, it follows that $H_i \cap S_j = \{x_j, z_j\}$. So, without loss of generality, $y_i = x_j$. But then since $H_i \cup \{y_j\}$ cannot induce a theta in G_{i-1}, (H_i, y_j) is a wheel in G_{i-1}, which contradicts the fact that P_i is a good ear. This completes the proof of Claim 1.

Claim 2: *T is chordal and $\omega(T) \le 6$.*

Proof of Claim 2: By Claim 1, it suffices to show that $T[H]$, and, for $1 \le i \le q$, $T[V(P_i) \cup \{y_i\}]$ are all chordal and have clique number at most 6. Let $G_0 = G[H]$, and observe that,
(*) since G is triangle-free, for every $1 \le i \le q$, every interior vertex of P_i has at most one neighbor in G_{i-1}.

Let $H = v_1, \ldots, v_k, v_1$, and without loss of generality we assume that $u = v_1$ and $v = v_2$. Suppose C is a hole contained in $T[H]$. Since, by construction, $\{u, v\}$ is complete to $V(H) \setminus \{u, v\}$, $V(C) \cap \{u, v\} = \emptyset$. Let v_i be the smallest-indexed vertex of C. So $i \ge 3$. It follows that the edges of C are either edges of H or are of type 1 and hence, by the above observation (*), are short chords of H. It follows by the choice of v_i that v_{i+1} and v_{i+2} are the neighbors of v_i in C. But then $v_{i+1}v_{i+2}$ is a chord of C, a contradiction. Therefore $T[H]$ is chordal. Since the edges of $T[V(H) \setminus \{u, v\}]$ are either edges of H or short chords of H, $\omega(T[V(H) \setminus \{u, v\}]) \le 3$, and hence $\omega(T[V(H)]) \le 5$.

Now consider an ear P_i. Suppose that $T[V(P_i) \cup \{y_i\}]$ contains a hole C. By construction, S_i is a clique of $T[V(P_i) \cup \{y_i\}]$ that is complete to $V(P_i) \setminus \{x_i, z_i\}$, and hence $V(C) \cap S_i = \emptyset$. So $V(C) \subseteq V(P_i) \setminus \{x_i, z_i\}$. Recall that P_i is a chordless path in G, and that every edge of $T[V(P_i) \setminus \{x_i, z_i\}]$ that is not an edge of P_i is of type 1. Let $x_j z_j$ be such a type 1 edge, and suppose that y_j is not a vertex of P_i. Then $j > i$. By the above observation (*), y_j cannot be an interior vertex of P_k where $k > i$. So y_j is a vertex of G_{i-1}, and since the only vertex of G_{i-1} that can be adjacent in G_i to two interior vertices of P_i is y_i, it follows that $y_j = y_i$. Let H' be the hole obtained by augmenting H_i with P_i. Then the wheel (H', y_j) is contained in G_{j-1} and contradicts P_j being a good ear. Therefore $y_j \in V(P_i)$, and so every type 1 edge of $T[V(P_i) \setminus \{x_i, z_i\}]$ is a short chord of P_i. This contradicts the assumption that C is a hole of $T[V(P_i) \setminus \{x_i, z_i\}]$. Hence, $T[V(P_i) \cup \{y_i\}]$ is chordal. Since $T[V(P_i) \setminus \{x_i, z_i\}]$ consists of edges of P_i and short chords of P_i, it follows that $\omega(T[V(P_i) \setminus \{x_i, z_i\}]) \le 3$, and therefore $\omega([V(P_i) \cup \{y_i\}]) \le 6$. This completes the proof of Claim 2.

By Claim 2, it follows that there is a triangulation of G^* with clique number at most 6, and hence the treewidth of G^* is at most 5. $\square$

As mentioned above, Corneil and Rotics [15] proved that the clique-width of a graph G is at most $3 \times 2^{tw(G)-1}$, and so the following is a direct corollary of Theorem 5.1.

Corollary 5.2 *Triangle-free odd-signable graphs have clique-width at most 48.*

Theorem 5.3 *If G is (cap, 4-hole)-free odd-signable graph with no clique cutset, then G has clique-width at most 48.*

Proof: Let G be a (cap, 4-hole)-free odd-signable graph with no clique cutset. Let U be the set of universal vertices. We may assume that $G \setminus U$ contains a hole, since otherwise $G \setminus U$ is a clique and so its clique-width is 2. Let F be a maximal induced subgraph of $G \setminus U$ that is triangle-free and has no clique cutset. By Theorem 3.6, $G \setminus U$ is obtained from F by substituting cliques for vertices of F. Since F is triangle-free odd-signable, it follows from Corollary 5.2 that the clique-width of F is at most 48. Substituting a graph G_2 for a vertex of a graph G_1 gives a graph with clique-width at most the maximum of the clique-widths of G_1 and G_2 [19], [22]. A clique of size at least 2 has clique-width 2. Thus it follows that $G \setminus U$ has clique-width at most 48. Adding a universal vertex to a graph with at least one edge does not change the clique-width. Thus G has clique-width at most 48. □

Theorem 5.4 *If G is a (cap, 4-hole)-free odd-signable graph with no clique cutset, then G has treewidth at most* $6\omega(G) - 1$.

Proof: Let G be a (cap, 4-hole)-free odd-signable graph with no clique cutset. We may assume that G contains a hole, since otherwise G is a clique and so the treewidth of G is $|V(G)| - 1 = \omega(G) - 1$. Let U be the set of universal vertices of G, and note that $G \setminus U$ has no clique cutset. Let F be a maximal induced subgraph of $G \setminus U$ that is triangle-free and has no clique cutset. By Theorem 3.6, $G \setminus U$ is obtained from F by, for each vertex v, substituting a clique K_v. Since F is triangle-free odd-signable, it follows from Theorem 5.1 that the treewidth of F is at most 5. In particular, there is a triangulation T of F with clique number at most 6. We can obtain a triangulation T' of $G \setminus U$ by substituting the cliques K_v for the vertices v of T. Each of these cliques K_v has size at most $\omega(G) - |U|$, so the size of a largest clique in T' is at most $6(\omega(G) - |U|)$. We obtain a triangulation T'' of G by adding to T' the clique U and joining every vertex of U to every vertex of T'. The largest clique in T'' has size at most $6(\omega(G) - |U|) + |U| = 6\omega(G) - 5|U| \leq 6\omega(G)$. Thus G has treewidth at most $6\omega(G) - 1$. □

6 Algorithms for coloring and maximum weight stable set

In this section, we give polynomial-time algorithms for maximum weight stable set, q-coloring (that is, coloring with a fixed number q of colors), and chromatic number restricted to (cap, 4-hole)-free odd-signable graphs (and in particular, (cap, even hole)-free graphs). Our algorithms will take the following general approach.

1. Decompose the input graph G via clique cutsets into subgraphs that do not contain clique cutsets. These subgraphs are called *atoms.*
2. Find the solution for each atom using Theorems 3.6, 5.1, 5.3, 5.4.
3. Combine solutions to atoms along the clique cutsets to obtain a solution for G.

6.1 Clique cutset decomposition

Let $k \geq 1$ be a fixed integer. Tarjan [37] observed that G is k-colorable if and only if each atom of G is k-colorable. This implies that if one can solve q-coloring or chromatic number for atoms, then one can also solve these problems for G. It is straightforward to check that once a k-coloring of each atom is found, then it takes $O(n^2)$ time to combine these colorings to obtain a k-coloring of G.

In a slightly more complicated fashion, Tarjan [37] showed that once the maximum weight stable set problem is solved for atoms, one can solve the problem for G. Let $G = (V, E)$ be a graph with a weight function $w : V \to \mathbb{R}$. For a given subset $S \subseteq V$, we let $w(S) = \sum_{v \in S} w(v)$, and denote the maximum weight of a stable set of G by $\alpha_w(G)$. Suppose that G is decomposed into A and B via a clique cutset S, where A is an atom. We now explain Tarjan's approach. To compute a stable set of weight $\alpha_w(G)$, we do the following.

(i) Compute a maximum weight stable set I' of $A \setminus S$.

(ii) For each vertex $v \in S$, compute a maximum weight stable set I_v of $A \setminus N[v]$.

(iii) Re-define the weight of $v \in S$ as $w'(v) = w(v) + w(I_v) - w(I')$.

(iv) Compute the maximum weight stable set I'' of B with respect to the new weight w'. If $I'' \cap S = \{v\}$, then let $I = I_v \cup I''$; otherwise let $I = I' \cup I''$.

It is easy to see that $\alpha_w(G) = w(I)$. This divide-and-conquer approach can be applied top-down on T_G to obtain a solution for G by solving $O(n^2)$ subproblems on induced subgraphs of atoms, as there are $O(n)$ decomposition steps and each step amounts to solving $O(n)$ subproblems as explained in (i)-(iv).

Therefore, it suffices to explain how to solve coloring and maximum weight stable set for atoms of (cap, 4-hole)-free odd-signable graphs. We do this below.

6.2 Skeleton

We say that two vertices u and v of a graph G are *true twins* if $N_G[u] = N_G[v]$. In particular, any pair of true twins are adjacent. It is clear that the binary relation on $V(G)$ defined by being true twins is an equivalence relation and therefore $V(G)$ can be partitioned into equivalence classes of true twins. Let U be the set of universal vertices of graph G; the *skeleton* of G is the subgraph of $G \setminus U$ induced by a set of vertices consisting of one vertex from each equivalence class of true twins.

Let G be a (cap, 4-hole)-free odd-signable graph without clique cutsets. By Theorem 3.6, G is obtained from a (triangle, 4-hole)-free induced subgraph F that has no clique cutset by first blowing up vertices $v \in V(F)$ into cliques K_v, and then adding a (possibly empty) universal clique U. The cliques K_v $(v \in V(F))$ and U are equivalence classes of true twins, and F is the skeleton of G.

Our algorithm relies on finding equivalence classes efficiently. The following theorem is left as an exercise in [36]. We give a proof. We say that a vertex u *distinguishes* vertices v and w if u is adjacent to exactly one of v and w.

Theorem 6.1 *Given a graph G with n vertices and m edges, one can find all equivalence classes of true twins in $O(n + m)$ time.*

Proof: Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. We think of each vertex being adjacent to itself, and consequently any vertex v does not distinguish v and any neighbor of v. The idea is to start with the trivial partition $\mathcal{P}_0 = \{V(G)\}$ and obtain a sequence of partitions $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of $V(G)$ such that $\mathcal{P}_i$ is a refinement of $\mathcal{P}_{i-1}$ and is obtained as follows: for each set $S \in \mathcal{P}_{i-1}$, we partition S into two subsets $S' = S \cap N[v_i]$ and $S'' = S \setminus S'$, and $\mathcal{P}_i = \cup_{S \in \mathcal{P}_{i-1}} \{S', S''\}$. It can be easily proved by induction that for each i it holds that (i) any pair of vertices in a set $S \in \mathcal{P}_i$ are not distinguished by any of $v_1, \ldots, v_i$; (ii) vertices from different sets in $\mathcal{P}_i$ are distinguished by one of $v_1, \ldots, v_i$. Therefore, the equivalence classes of true twins are exactly the non-empty sets in $\mathcal{P}_n$.

It remains to show that this can be implemented in $O(n+m)$ time. In the algorithm, we do not actually maintain the sets in a partition. Instead, we use an array $s[v_i]$ to keep track of which subset v_i belongs to. Initially, we set $s[v_i] = 0$ for all i and this takes $O(n)$ time. Then we do the following: for each $1 \leq i \leq n$, we set $s[u] = s[u] + 2^{i-1}$ for each $u \in N[v_i]$. Clearly, this takes $\sum_{v_i} O(d(v_i)) = O(m)$ time. In the end, we group vertices with the same s-value by scanning the array once and this takes $O(n)$ time. Therefore, the total running time is $O(n+m)$. □

In the following algorithms, we assume that G is the input graph. We first use Tarjan's algorithm to find T_G in $O(nm)$ time. For any atom A of G, we let n_A and m_A be the number of vertices and the number of edges of A, respectively. By Theorem 6.1 we can find the skeleton F of A, K_v and U in $O(n_A + m_A)$ time. Therefore, it takes $O(n)O(n+m) = O(nm)$ time to find skeletons for all atoms of G. So, we fix an atom A and assume that the skeleton F of A, K_v ($v \in V(F)$) and U are given.

6.3 Solving chromatic number using clique-width

It follows from Theorem 5.1 and Theorem 5.3 that F has treewidth at most 5 and clique-width at most 48. We first find a tree decomposition of F with width at most 5 in linear time by Bodlaendar [3], and then feed this decomposition into the algorithm of Espelage, Gurski, and Wanke [21] which outputs in linear time a k-expression of F for some constant k (k could be larger than 48). Then we construct from F, K_v ($v \in V(F)$) and U in linear time a k-expression of G [19]. Finally, we find the chromatic number of A in polynomial time by Kobler and Rotics [27]. We solve chromatic number for every atom of G in this way. The total running time is dominated by Kobler and Rotics's algorithm [27] which runs in $O(2^{3k+1}k^2n^{2^{2k+1}+1})$ time.

6.4 Solving q-coloring using treewidth

We first find the clique number $\omega(G)$ of G in $O(nm)$ time [1]. If $\omega(G) > q$, then G is not q-colorable, and we are done. Otherwise, $\omega(G) \leq q$ and so every atom A also has $\omega(A) \leq q$. By Theorem 5.4, the treewidth of A is at most $6q-1$. We then use Bodlaender's algorithm [3] to find a tree decomposition with width $6q-1$ in $O(n_A)$ time. Finally, q-coloring can be solved in $O(n_A)$ time for A [4, 17]. Since there are $O(n)$ atoms, the running time for find all colorings of atoms is $O(n)O(n) = O(n^2)$. Recall that combining colorings of atoms can also be done in $O(n^2)$ time, and so the total running time is dominated by finding T_G and skeletons, that is, $O(nm)$.

6.5 Solving maximum weight stable set using treewidth

For maximum weight stable set, we let $v' \in K_v$ be the vertex with maximum weight among vertices in K_v. Similarly, if $U \neq \emptyset$ then let $u' \in U$ be the vertex with largest weight among vertices in U. Let $F' = \{u'\} \cup \{v' : v \in V(F)\}$ if $U \neq \emptyset$, and $F' = \{v' : v \in V(F)\}$ if $U = \emptyset$. Note that F' is obtained from F by adding at most one universal vertex. Moreover, the maximum weight of a stable set in A equals the maximum weight of a stable set in F'. It follows from Theorem 5.1 that F has treewidth at most 5, and so F' has treewidth at most 6. Let S_A be the clique cutset used in the decomposition step that yields A, and $n'_A = |V(A \setminus S_A)|$. Recall that, for each atom A, we need to solve $O(n)$ subproblems on induced subgraphs of $A \setminus S_A$. Each such subproblem can be solved in $O(n'_A)$ time by first finding a tree decomposition with width at most 6 in $O(n'_A)$ time by Bodlaender [3], and then solving maximum weight stable set in $O(n'_A)$ time [4]. Note that for two different atoms A and B, the subgraphs of A for which the subproblems need to be solved are vertex-disjoint from the subgraphs of B for which the subproblems need to be solved. This implies that it takes $O(n) \sum_A n_A = O(n^2)$ time to solve all these subproblems, where the summation goes over all atoms of G. So, the total running time is dominated by finding T_G and skeletons, that is, $O(nm)$.

An important feature of our algorithms is that they are *robust* in the sense that we do not need to assume that the input graph is (cap, 4-hole)-free odd-signable. Our algorithms either report that the graph is not (cap, 4-hole)-free odd-signable or solve the problems (in which case the input graph may or may not be (cap, 4-hole)-free odd-signable): for any input graph G, we find the skeleton F of each atom A of G and test if F has treewidth at most 5. If for some atom, the answer is no, then G is not (cap, 4-hole)-free odd signable by Theorem 5.1; otherwise we use the above algorithms to solve coloring or maximum weight stable set. Note that we can test if the treewidth is at most 5 in linear time [3], so this does not increase the complexity of the algorithms.

6.6 Recognition

Even-hole-free graphs were first shown to be recognizable in polynomial time in [13]. Currently, the fastest known recognition algorithm for this class has complexity $O(n^{11})$ [9]. In [12], an $O(n^4)$ algorithm is given for recognizing triangle-free odd-signable graphs (and in particular (triangle, even hole)-free graphs). In [11] an $O(n^6)$ algorithm is given for recognizing cap-free odd-signable graphs (and in particular (cap, even hole)-free graphs). We now show how to do this in $O(n^5)$-time.

Lemma 6.2 *There is an $O(nm^2)$ time algorithm to decide if a graph contains a cap.*

Proof: We first guess an edge $e = uv$ and a vertex w such that w is the vertex that is adjacent to u and v which are in a hole not containing w. Clearly, there are m choices for e and at most n choices for w. We then test if u and v are in the same component of $G' = G \setminus ((N[w] \setminus \{u, v\}) \cup (N(u) \cap N(v)) \cup \{e\})$. This can be done in $O(n + m)$ time using breadth-first search. Therefore, the total running time is $O(m)O(n)O(n + m) = O(nm^2)$. The correctness follows from the fact that if there is a cap that consists of a hole H going through u and v, and the vertex w that is not on H, then there must exist a uv-path in G'. □

Lemma 6.3 *Let G be a graph that contains a universal vertex u. Then G is odd-signable if and only if $G \setminus \{u\}$ is even-hole-free.*

Proof: Suppose that G is odd-signable. Then G does not contain thetas, prisms or even wheels by Theorem 2.1. The fact that u is universal implies that $G \setminus \{u\}$ is even-hole-free. Conversely, if $G \setminus \{u\}$ is even-hole-free, then clearly G has no even wheels. Furthermore, $G \setminus \{u\}$ has no prisms and thetas, as it is even-hole-free. Note that thetas and prisms do not contain universal vertices, and so adding a universal vertex to $G \setminus \{u\}$ does not create a theta or prism. This shows that G is odd-signable by Theorem 2.1. □

Theorem 6.4 *There exists an $O(n^5)$ time algorithm to decide if a graph is (cap, 4-hole)-free odd-signable (resp. (cap, even hole)-free).*

Proof: Let G be a graph. We first test if G contains a 4-hole using brute force, and this takes $O(n^4)$ time. If G contains a 4-hole, then we stop. Therefore, we now assume that G is 4-hole-free. Secondly, we apply Lemma 6.2 to see if G contains a cap in $O(nm^2) = O(n^5)$ time. If G contains a cap, we stop. So, we may assume that G is (cap, 4-hole)-free.

We then apply Tarjan's algorithm to find the clique cutset decomposition tree T_G in $O(nm)$ time. It is easy to see that G is odd-signable if and only if each atom is. For each atom A, we find its skeleton F, K_v for $v \in V(F)$ and U in $O(n+m)$ time by Theorem 6.1. If $U = \emptyset$, then A is odd-signable if and only if F is odd-signable, since adding twin vertices preserves being odd-signable; if $U \neq \emptyset$, i.e., A contains a universal vertex, then it follows from Lemma 6.3 that A is odd-signable if and only if F is even-hole-free. We finally apply the $O(n^4)$ time recognition algorithm from [12] for triangle-free odd-signable graphs or (triangle, even hole)-free graphs to F depending on whether U is empty or not. If the algorithm returns no for the skeleton F of some atom, then G is not odd-signable; otherwise G is odd-signable. The running time for testing all atoms is $O(n)O(n^4) = O(n^5)$. Therefore, the total running time for recognizing (cap, 4-hole)-free odd-signable graphs is $O(n^5)$. Similarly, (cap, even hole)-free graphs can be recognized with the same time complexity. □

7 Open Problems

The bound given by Corollary 4.2 is attained by odd holes and the Hajós graph (see Figure 2). Note that these graphs have clique number at most 3. For graphs with large clique number, we do not have an example showing that the bound is tight. Nevertheless, the optimal constant is at least $\frac{5}{4}$. For any integer $k \geq 1$, let G_k be the graph obtained from a 5-hole by substituting a clique of size $2k$ for each vertex of the 5-hole. Clearly, $|V(G_k)| = 10k$, $\alpha(G_k) = 2$ and $\omega(G_k) = 4k$. Hence, $\chi(G_k) \geq \frac{|V(G_k)|}{\alpha(G_k)} = 5k$. Moreover, it is easy to see that G_k does admit a $5k$-coloring. So, $\chi(G_k) = 5k = \frac{5}{4}\omega(G_k)$. A natural question is whether or not one can reduce the constant from $\frac{3}{2}$ to $\frac{5}{4}$.

Problem: *Is it true that $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$ for every (cap, even hole)-free graph G?*

It was shown in [10] that this is true for the class of (C_4, P_5)-free graphs, which is a subclass of the class of (cap, even hole)-free graphs.

Even-hole-free graphs are also known to be χ-bounded. In [2] it is shown that every even-hole-free graph has a vertex whose neighborhood is a union of two cliques. This implies

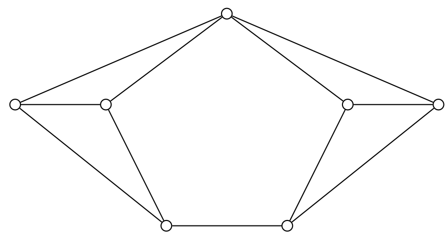

Figure 2: The Hajós graph.

that if G is even-hole-free, then $\chi(G) \leq 2\omega(G) - 1$. It remains open whether a better bound is possible.

The complexity of 3-coloring, q-coloring, and minimum coloring is unknown for even-hole-free graphs, 4-hole-free odd-signable graphs, and odd-signable graphs. Polynomial-time algorithms for minimum coloring have been given for (diamond, even hole)-free graphs [26] and (pan, even hole)-free graphs [8].

The clique covering problem is to find a minimum number of cliques which partition the vertices of a graph. This problem is the same as finding a minimum coloring of the complementary graph. The complexity of this problem is unknown for the following classes of graphs: (cap, even hole)-free graphs, (cap, 4-hole)-free odd-signable graphs, 4-hole-free odd-signable graphs, even-hole-free graphs, and odd-signable graphs.

Acknowledgement. We would like to thank Haiko Müller for fruitful discussions, and to thank Jerry Spinrad for pointing to us the idea of finding true twins in $O(n + m)$ time.

References

[1] P. Aboulker, P. Charbit, N. Trotignon and K. Vušković, *Vertex elimination orderings for hereditary graph classes* Discrete Mathematics 338 (2015) no. 5, 825-834.

[2] L. Addario-Berry, M. Chudnovsky, F. Havet, B. Reed and P. Seymour, *Bisimplicial vertices in even-hole-free graphs*, Journal of Combinatorial Theory B 98 (2008) 1119-1164.

[3] H. L. Bodlaender, *A linear-time algorithm for finding tree-decompositions of small treewidth*, SIAM J. Comput. 25 (1996) no. 6, 1305-1317.

[4] H. L. Bodlaender and A. M. C. A. Koster, *Combinatorial optimization on graphs of bounded treewidth*, The Computer Journal 51 (2008) no. 3, 255-269.

[5] M. Burlet and J. Fonlupt, *Polynomial algorithm to recognize a Meyniel graph*, Discrete Mathematics 21 (1984) 225-252.

[6] K. Cameron and J. Edmonds, *An algorithm for finding a strong stable set or a Meyniel obstruction in any graph*, Discrete Mathematics and Theoretical Computer Science Proceedings (2005) 203-206.

[7] K. Cameron, B. Lévêque and F. Maffray, *Coloring vertices of a graph or finding a Meyniel obstruction*, Theoret. Comput. Sci. 428 (2012) 10-17.

[8] K. Cameron, S. Chaplick and C. T. Hoàng, *On the structure of pan, even hole-free graphs*, arXiv:1508.03062 [cs.DM], 2015, to appear in Journal of Graph Theory.

[9] H.-C. Chang and H.-I. Lu, *A faster algorithm for recognising even-hole-free graphs*, Journal of Combinatorial Theory B 113 (2015) 141-161.

[10] S.A. Choudum, T. Karthick, M. A. Shalu, *Perfectly coloring and linearly χ-bound P_6-free graphs*, Journal of Graph Theory (2007) 293-306.

[11] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, *Even and odd holes in cap-free graphs*, Journal of Graph Theory 30 (1999) 289-308.

[12] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, *Triangle-free graphs that are signable without even holes*, Journal of Graph Theory 34 (2000) 204-220.

[13] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, *Even-hole-free graphs, Part II: Recognition algorithm*, Journal of Graph Theory 40 (2002) 238-266.

[14] M. Conforti, B. Gerards and K. Pashkovich, *Stable sets and graphs with no even hole*, Math. Program., Ser. B 153 (2015) 13-39.

[15] D. G. Corneil and U. Rotics, *On the relationship between clique-width and treewidth*, SIAM J. Comput. 34 (2005) no. 4, 825-847.

[16] G. Cornuéjols and W. H. Cunningham, *Compositions for perfect graphs*, Discrete Mathematics 55 (1985) 245-254.

[17] B. Courcelle, *The monadic second-order logic of graphs. i. Recognizable sets of finite graphs*, Information and computation 85 (1990) no. 1, 12-75.

[18] B. Courcelle, J. Makowsky, and U. Rotics, *Linear time solvable optimization problems on graphs of bounded clique-width*, Theory of Computing Systems 33 (2000) no. 2, 125-150.

[19] B. Courcelle and S. Olariu, *Upper bounds to the clique width of graphs*, Discrete Appl. Math. 101 (2000) no. 1-3, 77-114.

[20] M. V. G. da Silva and K. Vušković, *Triangulated neighborhoods in even-hole-free graphs*, Discrete Mathematics 307 (2007) 1065-1073.

[21] W. Espelage, F. Gurski, and E. Wanke, *Deciding clique-width for graphs of bounded tree-width*, J. Graph Algorithms Appl. 7 (2003) no. 2, 141-180.

[22] F. Gurski, *Graph operations on clique-width bounded graphs*, arXiv:cs/0701185v3 [cs.DS], 2016.

[23] A. Hertz, *A fast algorithm for colouring Meyniel graphs*, Journal of Combinatorial Theory B 50 (1990) 231-240.

[24] C. T. Hoàng, *On a conjecture of Meyniel*, J. Comb. Th. B 42 (1987) 302-312.

[25] J. B. Kelly and L. M. Kelly, *Paths and circuits in critical graphs*, Amer. J. Math. 76 (1954) 786-792.

[26] T. Kloks, H. Müller and K. Vušković, *Even-hole-free graphs that do not contain diamonds: a structure theorem and its consequences*, J. Combin. Theory Ser. B 99 (2009) no. 5, 733800.

[27] D. Kobler and U. Rotics, *Edge dominating set and colorings on graphs with fixed clique-width*, Discrete Appl. Math. 126 (2003) no. 2-3, 197-221.

[28] S. E. Markosjan and I. A. Karapetjan, *Perfect graphs*, Akad. Nauk Armjan. SSR. Dokl. 63 (1976) 292-296.

[29] S. E. Markossian, G. S. Gasparian and B. A. Reed, *β-perfect graphs*, Journal of Combinatorial Theory B 67 (1996) 1-11.

[30] H. Meyniel, *On the perfect graph conjecture*, Discrete Mathematics 16 (1976) 339-342.

[31] S. Oum, *Approximating rank-width and clique-width quickly*, ACM Trans. Algorithms 5 (2009) no. 1, Art. 10, 20 pp.

[32] S. Oum and P. D. Seymour, *Approximating clique-width and branchwidth*, Journal of Combinatorial Theory, Series B, 96 (2006) no. 4, 514-528.

[33] F. Roussel and I. Rusu, *Holes and dominoes in Meyniel graphs*, Internat. J. Found. Comput. Sci. 10 (1999) 127-146.

[34] F. Roussel and I. Rusu, *An $O(n^2)$ algorithm to color Meyniel graphs*, Discrete Mathematics (2001) 107-123.

[35] A. Silva, A. A. da Silva and C. Linhares Sales, *A bound on tree width of planar even-hole-free graphs*, Discrete Applied Mathematics 158 (12) 1229-1239.

[36] J. P. Spinrad, *Efficient Graph Representations*, American Mathematical Society, 2003.

[37] R. E. Tarjan, *Decomposition by clique separators*, Discrete Mathematics (1985) 221-232.

[38] K. Truemper, *Alpha-balanced graphs and matrices and GF(3)-representability of matroids*, Journal of Combinatorial Theory B 32 (1982) 112-139.

[39] K. Vušković, *Even-hole-free graphs: A survey*, Appl. Anal. Discrete Math. 4 (2010) 219-240.