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Triangulated neighborhoods in even-hole-free graphs☆

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Abstract

An even-hole-free graph is a graph that does not contain, as an induced subgraph, a chordless cycle of even length. A graph is triangulated if it does not contain any chordless cycle of length greater than three, as an induced subgraph. We prove that every even-hole-free graph has a node whose neighborhood is triangulated. This implies that in an even-hole-free graph, with n nodes and m edges, there are at most $n + 2m$ maximal cliques. It also yields an $O(n^2m)$ algorithm that generates all maximal cliques of an even-hole-free graph. In fact these results are obtained for a larger class of graphs that contains even-hole-free graphs.

Keywords: Even-hole-free graphs; Triangulated graphs; Structural characterization; Generating all maximal cliques

1. Introduction

We say that a graph G *contains* a graph H, if H is isomorphic to an induced subgraph of G. A graph G is *H-free* if it does not contain H. A *hole* is a chordless cycle of length at least four. A hole is *even* (resp. *odd*) if it contains even (resp. odd) number of nodes. An *n-hole* is a hole of length n. A graph is said to be *triangulated* if it does not contain any hole.

We *sign* a graph by assigning 0, 1 weights to its edges in such a way that, for every triangle in the graph, the sum of the weights of its edges is odd. A graph G is *odd-signable* if there is a signing of its edges so that, for every hole in G, the sum of the weights of its edges is odd. Clearly every even-hole-free graph is odd-signable, since we can get a correct signing by assigning a weight of 1 to every edge of the graph.

The graphs that are odd-signable and do not contain a 4-hole are studied in [7], where a decomposition theorem is proved for them. This decomposition theorem is used in [8] to obtain a polynomial time recognition algorithm for even-hole-free graphs.

For $x \in V(G)$, $N(x)$ denotes the set of nodes of G that are adjacent to x, and $N[x] = N(x) \cup \{x\}$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph of G induced by V'. For $x \in V(G)$, the graph $G[N(x)]$ is called the *neighborhood* of x.

The main result of this paper is the following structural characterization of odd-signable graphs that do not contain a 4-hole.

Theorem 1.1. *Every* 4*-hole-free odd-signable graph has a node whose neighborhood is triangulated.*

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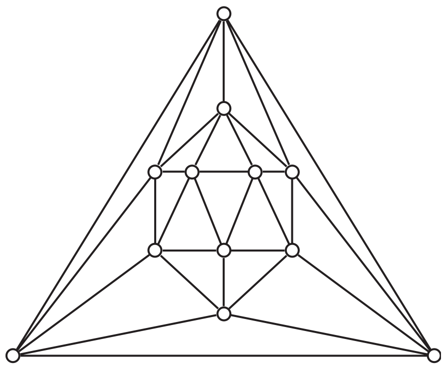

Fig. 1. A 4-hole-free graph that has no vertex whose neighborhood is triangulated.

Exactly the same characterization of 4-hole-free Berge graphs (i.e. graphs that do not contain a 4-hole nor an odd hole) is obtained by Parfenoff et al. [15]. Note that 4-hole-free graphs in general need not have this property, see Fig. 1.

A graph is *Berge* if it does not contain an odd hole nor the complement of an odd hole. A *square*-$3PC(\cdot,\cdot)$ is a graph that consists of three paths between two nodes such that any two of the paths induce a hole, and at least two of the paths are of length 2. A graph G is *even-signable* if there is a signing of its edges so that for every hole in G, the sum of the weights of its edges is even. In [13] Maffray et al. show that every square-$3PC(\cdot,\cdot)$-free even-signable graph has a node whose neighborhood does not contain a long hole (where a *long hole* is a hole of length greater than 4). This result is used in [13] to obtain a combinatorial algorithm of complexity $\mathcal{O}(n^7)$ for finding a clique of maximum weight in square-$3PC(\cdot,\cdot)$-free Berge graphs. Note that this class of graphs generalizes both 4-hole-free Berge graphs and claw-free Berge graphs (where a *claw* is a graph on nodes x, a, b, c with three edges xa, xb, xc). We show in this paper that key ideas from [13] extend to 4-hole-free odd-signable graphs.

Using Theorem 1.1 one can obtain an efficient algorithm for generating all the maximal cliques in 4-hole-free odd-signable graphs (and in particular even-hole-free graphs). This we describe in Section 2. Theorem 1.1 is proved in Section 3.

Recently Addario-Berry et al. [1] have proved a stronger property of even-hole-free graphs, namely that every even-hole-free graph has a bisimplicial vertex (i.e. a vertex whose neighborhood partitions into two cliques). This characterization immediately yields that for an even-hole-free graph G, $\chi(G) \leqslant 2\omega(G)-1$, where $\chi(G)$ is the chromatic number of G and $\omega(G)$ is the size of the largest clique in G (observe that if v is a bisimplicial vertex of G, then its degree is at most $2\omega(G)-2$, and hence G can be colored with at most $2\omega(G)-1$ colors). The two characterizations of even-hole-free graphs were discovered independently and at about the same time. The proof of the characterization in [1] is over 40 pages long. Our weaker characterization is enough to obtain an efficient algorithm for generating all maximal cliques of even-hole-free graphs, and its proof is very short.

2. Generating all the maximal cliques of a 4-hole-free odd-signable graph

For a graph G let k denote the number of maximal cliques in G, n the number of nodes in G and m the number of edges of G. Farber [10] shows that there are $\mathcal{O}(n^2)$ maximal cliques in any 4-hole-free graph. Tsukiyama et al. [19] give an $\mathcal{O}(nmk)$ algorithm for generating all maximal cliques of a graph, and Chiba and Nishizeki [2] improve this complexity to $\mathcal{O}(m^{1.5}k)$. The complexity is further improved for dense graphs by the $\mathcal{O}(M(n)k)$ algorithm of Makino and Uno [14], where $M(n)$ denotes the time needed to multiply two $n \times n$ matrices. Note that Coppersmith and Winograd show that matrix multiplication can be done in $\mathcal{O}(n^{2.376})$ time [9]. So one can generate all the maximal cliques of a 4-hole-free graph in time $\mathcal{O}(m^{1.5}n^2)$ or $\mathcal{O}(n^{4.376})$.

We now show that Theorem 1.1 implies that there are at most $n+2m$ maximal cliques in a 4-hole-free odd-signable graph, and it yields an algorithm that generates all the maximal cliques of a 4-hole-free odd-signable graph in time $\mathcal{O}(n^2m)$. In particular, in a weighted graph, a maximum weight clique can be found in time $\mathcal{O}(n^2m)$.

Let $\mathscr{C}$ be any class of graphs closed under taking induced subgraphs, such that for every G in $\mathscr{C}$, G has a node whose neighborhood is triangulated. Consider the following algorithm for generating all maximal cliques of graphs in $\mathscr{C}$.

Find a node x_1 of G whose neighborhood is triangulated (if no such node exists, G is not in $\mathscr{C}$, or in particular, G is not 4-hole-free odd-signable graph by Theorem 1.1). Let $G_1 = G[N[x_1]]$ and $G^1 = G\backslash\{x_1\}$. Every maximal clique of G belongs to G_1 or G^1. Recursively construct triangulated graphs $G_1, \ldots, G_n$ as follows. For $i \geqslant 2$, find a node x_i of G^{i-1} whose neighborhood is triangulated and let $G_i = G[N_{G^{i-1}}[x_i]]$ and $G^i = G^{i-1}\backslash\{x_i\} = G\backslash\{x_1, \ldots, x_i\}$.

Clearly every maximal clique of G belongs to exactly one of the graphs $G_1, \ldots, G_n$. A triangulated graph on n vertices has at most n maximal cliques [11]. So for $i = 1, \ldots, n$, graph G_i has at most $1 + d(x_i)$ maximal cliques (where $d(x)$ denotes the degree of vertex x). It follows that the number of maximal cliques of G is at most $\sum_{i=1}^{n}(1+d(x_i)) = n+2m$.

Checking whether a graph is triangulated can be done in time $\mathscr{O}(n + m)$ (using lexicographic breadth-first search [16]). So finding a vertex with triangulated neighborhood can be done in time $\mathscr{O}(\sum_{x\in V(G)}(d(x)+m)) = \mathscr{O}(nm)$. Hence, constructing the graphs $G_1, \ldots, G_n$ takes time $\mathscr{O}(n^2m)$.

Generating all maximal cliques in a triangulated graph can be done in time $\mathscr{O}(n + m)$ (see, for example, [12]). Hence the overall complexity of generating all maximal cliques in a 4-hole-free odd-signable graph is dominated by the construction of the sequence $G_1, \ldots, G_n$, i.e. it is $\mathscr{O}(n^2m)$.

Note that this algorithm is *robust* in Spinrad's sense [17]: given any graph G, the algorithm either verifies that G is not in $\mathscr{C}$ (or in particular that G is not a 4-hole-free odd-signable graph) or it generates all the maximal cliques of G. Note that, when G is not in $\mathscr{C}$, the algorithm might still generate all the maximal cliques of G.

3. Proof of Theorem 1.1

For a graph G, let $V(G)$ denote its node set. For simplicity of notation we will sometimes write G instead of $V(G)$, when it is clear from the context that we want to refer to the node set of G. Also a singleton set $\{x\}$ will sometimes be denoted with just x. For example, instead of "$u \in V(G)\backslash\{x\}$", we will write "$u \in G\backslash x$".

Let x, y be two distinct nodes of G. A $3PC(x, y)$ is a graph induced by three chordless x, y-paths, such that any two of them induce a hole. We say that a graph G contains a $3PC(\cdot, \cdot)$ if it contains a $3PC(x, y)$ for some $x, y \in V(G)$. $3PC(\cdot, \cdot)$'s are also known as *thetas* (for example in [5]).

Let $x_1, x_2, x_3, y_1, y_2, y_3$ be six distinct nodes of G such that $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ induce triangles. A $3PC(x_1x_2x_3, y_1y_2y_3)$ is a graph induced by three chordless paths $P_1 = x_1, \ldots, y_1$, $P_2 = x_2, \ldots, y_2$ and $P_3 = x_3, \ldots, y_3$, such that any two of them induce a hole. We say that a graph G contains a $3PC(\Delta, \Delta)$ if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. $3PC(\Delta, \Delta)$'s are also known as *prisms* (for example in [4]).

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, \ldots, x_n$. Node x is the *center* of the wheel. We say that the wheel (H, x) is *even* when n is even.

It is easy to see that even wheels, $3PC(\cdot, \cdot)$'s and $3PC(\Delta, \Delta)$'s cannot be contained in even-hole-free graphs. In fact they cannot be contained in odd-signable graphs. The following characterization of odd-signable graphs, given in [6], states that the converse is also true. It is in fact an easy consequence of a theorem of Truemper [18].

Theorem 3.1. *A graph is odd-signable if and only if it does not contain an even wheel, a* $3PC(\cdot, \cdot)$ *nor a* $3PC(\Delta, \Delta)$.

The fact that odd-signable graphs do not contain even wheels, $3PC(\cdot, \cdot)$'s and $3PC(\Delta, \Delta)$'s will be used throughout the rest of the paper.

In the next three lemmas we assume that G is a 4-hole-free odd-signable graph, x a node of G that is not adjacent to every other node of G, C_1 a connected component of $G\backslash N[x]$, and H a hole of $N(x)$. Note that H is an odd hole, else (H, x) is an even wheel.

Lemma 3.2. *If node u of C_1 has a neighbor in H then u is one of the following two types*:

- *Type* 1: *u has exactly one neighbor in H.*
- *Type* 2: *u has exactly two neighbors in H, and they are adjacent.*

Proof. If u has two nonadjacent neighbors a and b in H, then $\{a, b, u, x\}$ induces a 4-hole. □

Let T^3 be a graph on 3 nodes that has exactly one edge.

Let x_1, x_2, x_3, y be four distinct nodes of G such that x_1, x_2, x_3 induce a triangle. A $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P_1 = x_1, \ldots, y$, $P_2 = x_2, \ldots, y$ and $P_3 = x_3, \ldots, y$, such that any two of them induce a hole. We say that a graph G contains a $3PC(\Delta, \cdot)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$. $3PC(\Delta, \cdot)$'s are also known as *pyramids* (for example in [3]).

Lemma 3.3. *If H contains a T^3 all of whose nodes have neighbors in C_1, then C_1 contains a path P, of length greater than* 0, *such that $P \cup H$ induces a $3PC(\Delta, \cdot)$, and the nodes of H that have a neighbor in P induce a T^3.*

Proof. Let C be a smallest subset of C_1 such that $G[C]$ is connected and $H = h_1, \ldots, h_n, h_1$ contains a T^3 all of whose nodes have neighbors in C. W.l.o.g. h_1, h_2 and h_i, $3 < i < n$, have neighbors in C. Let $P = p_1, \ldots, p_k$ be a shortest path of C such that p_1 is adjacent to h_1 and p_k is adjacent to h_2. Note thatno intermediate node of P is adjacent to h_1 or h_2. Also possibly $k = 1$.

Claim 1. *No node of $\{h_4, \ldots, h_{n-1}\}$ has a neighbor in P.*

Proof of Claim 1. Suppose not. Then by minimality of C, h_i has a neighbor in P and w.l.o.g. no node of $\{h_{i+1}, \ldots, h_{n-1}\}$ has a neighbor in P. By Lemma 3.2, $p_1, p_k \notin N(h_i) \cap P$. In particular $k > 1$.

First suppose $N(h_n) \cap P \neq \emptyset$. By Lemma 3.2, $h_n p_k$ is not an edge. If $N(h_n) \cap P = p_1$ then $\{x, h_n, h_2, h_1\} \cup P$ induces an even wheel with center h_1. So h_n has a neighbor in $P \backslash \{p_1, p_k\}$. If $h_i h_n$ is not an edge, then since all of h_1, h_n, h_i have neighbors in $P \backslash p_k$, the minimality of C is contradicted. So $h_i h_n$ is an edge of G. But then all of h_i, h_n, h_2 have neighbors in $P \backslash p_1$ and the minimality of C is contradicted. So $N(h_n) \cap P = \emptyset$.

Let p_r be the node of P with highest index adjacent to h_i. Let H' be the hole induced by $\{h_i, \ldots, h_n, h_1, h_2, p_k, \ldots, p_r\}$. Since (H', x) cannot be an even wheel, it follows that $h_i, \ldots, h_n, h_1, h_2$ is an even subpath of H. Let p_s be the node of P with lowest index adjacent to h_i. Then $\{x, h_i, \ldots, h_n, h_1, p_1, \ldots, p_s\}$ induces an even wheel with center x. This completes the proof of Claim 1. $\square$

By Claim 1, h_i is not adjacent to a node of P. But h_i has a neighbor in C, and since C is connected, let $Q = q_1, \ldots, q_l$ be a chordless path in C such that q_1 is adjacent to h_i and q_l has a neighbor in P.

Claim 2. *No node of $\{h_4, \ldots, h_{n-1}\}$ has a neighbor in $(P \cup Q) \backslash q_1$.*

Proof of Claim 2. Suppose that some $h_j \in \{h_4, \ldots, h_{n-1}\}$ has a neighbor in $(P \cup Q) \backslash q_1$. Then all of h_1, h_2, h_j have neighbors in $(P \cup Q) \backslash q_1$, contradicting the minimality of C. This completes the proof of Claim 2. $\square$

Claim 3. *q_1 is of type* 1 *w.r.t. H.*

Proof of Claim 3. By Lemma 3.2 q_1 is of type 1 or type 2. Suppose q_1 is of type 2. We now prove that $N(q_1) \cap H$ is either $\{h_3, h_4\}$ or $\{h_{n-1}, h_n\}$. Assume not. Then q_1 is adjacent to neither h_3 nor h_n. W.l.o.g. $N(q_1) \cap H = \{h_i, h_{i-1}\}$ and $i \neq 4$. If $N(q_l) \cap P \neq p_1$, then $(P \cup Q) \backslash p_1$ is connected and all of h_i, h_{i-1}, h_2 have neighbors in it, contradicting the minimality of C. So $N(q_l) \cap P = p_1$. If $k > 1$, then all of h_i, h_{i-1}, h_1 have neighbors in $(P \cup Q) \backslash p_k$, contradicting the minimality of C. So $k = 1$, and hence by Lemma 3.2, $N(p_1) \cap H = \{h_1, h_2\}$. Since H is odd, the two subpaths of H, $h_2, \ldots, h_{i-1}$ and $h_i, \ldots, h_n, h_1$ have different parities. W.l.o.g. $h_2, \ldots, h_{i-1}$ is odd, i.e. i is even. By Claim 2, no node of $\{h_4, \ldots, h_{n-1}\}$ has a neighbor in $(P \cup Q) \backslash q_1$. If h_3 has no neighbor in Q then $Q \cup P \cup \{h_2, \ldots, h_{i-1}, x\}$ contains an even wheel with center x. So h_3 must have a neighbor in Q. But then h_i, h_{i-1}, h_3 all have neighbors in Q (note that $h_3 h_{i-1}$ is not an edge since $i - 1$ is odd greater than 3) contradicting the minimality of C. So $N(q_1) \cap H$ is either $\{h_3, h_4\}$ or $\{h_{n-1}, h_n\}$.

W.l.o.g. $N(q_1) \cap H = \{h_3, h_4\}$. If $N(q_l) \cap P \neq p_k$, then since all of h_1, h_3, h_4 have neighbors in $(P \cup Q) \backslash p_k$, the minimality of C is contradicted. So $N(q_l) \cap P = p_k$.

If $N(h_1) \cap Q \neq \emptyset$, then since all of h_1, h_3, h_4 have neighbors in Q, the minimality of C is contradicted. So $N(h_1) \cap Q = \emptyset$.

Now suppose that $N(h_n) \cap Q \neq \emptyset$. If $k > 1$, then since all of h_2, h_3, h_n have neighbors in $(P \cup Q) \backslash p_1$, the minimality of C is contradicted. So $k = 1$. Let q_r be the neighbor of h_n with highest index. If h_2 does not have a neighbor in

$q_r, q_{r+1}, \dots, q_l$, then $\{q_r, q_{r+1}, \dots, q_l, p_1, h_1, h_2, h_n, x\}$ induces an even wheel with center h_1. So $N(h_2) \cap Q \neq \emptyset$. But then since h_2, h_3, h_n have neighbors in Q, the minimality of C is contradicted. Therefore, $N(h_n) \cap Q = \emptyset$. So, by Claim 2, no node of $h_5, \dots, h_n, h_1$ has a neighbor in Q.

Suppose $N(h_2) \cap Q \neq \emptyset$. Let q_r be the neighbor of h_2 in Q with lowest index. Then $(H \backslash h_3) \cup \{x, q_1, \dots, q_r\}$ induces an even wheel with center x. Therefore, $N(h_2) \cap Q = \emptyset$. If $k > 1$, then $Q \cup (H \backslash h_3) \cup \{p_k, x\}$ induces an even wheel with center x. So $k = 1$. Let q_s be the node of Q with highest index adjacent to h_3. Then $\{p_1, q_s, \dots, q_l, h_1, h_2, h_3, x\}$ induces an even wheel with center h_2. This completes the proof of Claim 3. □

Claim 4. *$N(q_l) \cap P = p_1$ or p_k.*

Proof of Claim 4. Assume not. Then $k > 1$, and both $(P \cup Q) \backslash p_1$ and $(P \cup Q) \backslash p_k$ are connected. $N(h_1) \cap Q = \emptyset$, else all of h_1, h_2, h_i have neighbors in $(P \cup Q) \backslash p_1$, contradicting the minimality of C. Similarly, $N(h_2) \cap Q = \emptyset$.

We now show that h_3 has no neighbor in $P \cup Q$. Suppose it does. Then by Lemma 3.2, h_3 has a neighbor in $(P \cup Q) \backslash p_1$. If $i \neq 4$, then since all h_2, h_3, h_i have neighbors in $(P \cup Q) \backslash p_1$, the minimality of C is contradicted. So $i = 4$. If $N(h_3) \cap (P \cup Q) \neq p_k$, then all of h_1, h_3, h_4 have neighbors in $(P \cup Q) \backslash p_k$, contradicting the minimality of C. So $N(h_3) \cap (P \cup Q) = p_k$. But then $P \cup Q \cup \{h_2, h_3, h_4, x\}$ contains an even wheel with center h_3. Therefore, h_3 has no neighbor in $P \cup Q$, and similarly neither does h_n.

By minimality of C, $N(q_l) \cap P$ is either a single vertex or two adjacent vertices of P. If $N(q_l) \cap P = \{a, b\}$, where $ab \in E(G)$, then $P \cup Q \cup \{x, h_1, h_2, h_i\}$ induces a $3PC(q_l ab, x h_1 h_2)$. If $N(q_l) \cap P = \{a\}$, then $P \cup Q \cup \{h_1, h_2, \dots, h_i\}$ induces a $3PC(a, h_2)$. This completes the proof of Claim 4. □

By Claim 4, w.l.o.g. $N(q_l) \cap P = p_k$.

Claim 5. *h_1 does not have a neighbor in $(P \cup Q) \backslash p_1$.*

Proof of Claim 5. If $k > 1$, the claim follows from the minimality of C. Now suppose $k = 1$ and $N(h_1) \cap Q \neq \emptyset$. If h_2 has a neighbor in Q, then all of h_1, h_2, h_i have a neighbor in Q, contradicting the minimality of C. So h_2 does not have a neighbor in Q.

Suppose h_n has a neighbor in Q. Note that by Claim 3, such a neighbor is in $Q \backslash q_1$. Then h_3 cannot have a neighbor in Q, else all of h_n, h_1, h_3 have neighbors in Q, contradicting the minimality of C. But then $(Q \backslash q_1) \cup (H \backslash h_1) \cup \{x, p_1\}$ contains an even wheel with center x. So h_n does not have a neighbor in Q.

Suppose h_3 has a neighbor in Q. By Claim 3, such a neighbor is in $Q \backslash q_1$. Then $(Q \backslash q_1) \cup (H \backslash h_2) \cup x$ contains an even wheel with center x. So h_3 does not have a neighbor in Q.

Let H' be the hole induced by $\{p_1, h_2, \dots, h_i\} \cup Q$, and H'' the hole induced by $\{x, p_1, h_2, h_i\} \cup Q$. Then either (H', h_1) or (H'', h_1) is an even wheel. This completes the proof of Claim 5. □

Claim 6. *$N(h_n) \cap (P \cup Q) = \emptyset$.*

Proof of Claim 6. Assume not. If h_3 has a neighbor in $P \cup Q$ then, by Claim 3, all of h_2, h_3, h_n have a neighbor in $(P \cup Q) \backslash q_1$, contradicting the minimality of C. So $N(h_3) \cap (P \cup Q) = \emptyset$. Let R be a shortest path from h_2 to h_n in the graph induced by $P \cup (Q \backslash q_1) \cup \{h_2, h_n\}$. Then by Claims 2 and 3, $R \cup (H \backslash h_1) \cup x$ induces an even wheel with center x. This completes the proof of Claim 6. □

Claim 7. *$N(h_3) \cap (P \cup Q) = \emptyset$.*

Proof of Claim 7. Assume not. Let R be a shortest path from h_1 to h_3 in the graph induced by $(P \cup Q) \backslash q_1$. Then $R \cup (H \backslash h_2) \cup x$ induces an even wheel with center x. This completes the proof of Claim 7. □

If $k > 1$ then the graph induced by $H \cup Q \cup p_k$ contains a $3PC(h_2, h_i)$. So $k = 1$. By symmetry and Claim 5, h_2 does not have a neighbor in Q, and hence $P \cup Q \cup H$ induces a $3PC(\Delta, \cdot)$.

Lemma 3.4. *There exists a node of H that has no neighbor in C_1.*

Proof. Let $H = h_1, \ldots, h_n, h_1$ and suppose that every node of H has a neighbor in C_1. Recall that since (H, x) cannot be an even wheel, H is of odd length. So H contains a T^3 all of whose nodes have neighbors in C_1. By Lemma 3.3, C_1 contains a path $P = p_1, \ldots, p_k$, $k > 1$, such that $P \cup H$ induces w.l.o.g. a $3PC(h_1h_2p_k, h_i)$, $3 < i < n$. If i is odd, then $\{x, h_2, \ldots, h_i\} \cup P$ induces an even wheel with center x. So i is even.

Let $Q = q_1, \ldots, q_l$ be a path in C_1 defined as follows: q_1 is adjacent to $h_j \in H\backslash\{h_1, h_2, h_i\}$ where j is odd, q_l is adjacent to a node of P and no proper subpath of Q has this property. We may assume that P and Q are chosen so that $|P \cup Q|$ is minimized.

By the choice of P and Q, $N(q_l) \cap P$ is either one single vertex or two adjacent vertices of P, and h_j has no neighbor in $Q\backslash q_1$. Note that since n is odd, the two subpaths of H, $h_2, \ldots, h_i$ and $h_i, \ldots, h_n, h_1$ are both of even length, so we may assume w.l.o.g. that $2 < j < i$.

Claim 1. *At least one node of* $\{h_2, \ldots, h_{j-1}\}$ *(resp.* $\{h_{j+1}, \ldots, h_n\}$*) has a neighbor in* Q.

Proof of Claim 1. First suppose that no node of $H\backslash\{h_1, h_j\}$ has a neighbor in Q. Let p_s be the node of P with highest index adjacent to q_l. If $j > 3$, then $\{x, h_2, \ldots, h_j, p_s, \ldots, p_k\} \cup Q$ induces an even wheel with center x. So $j = 3$. If $N(h_1) \cap Q = \emptyset$ then $\{x, h_1, h_2, h_3, p_s, \ldots, p_k\} \cup Q$ induces an even wheel with center h_2. So $N(h_1) \cap Q \neq \emptyset$. Let q_r be the node of Q with lowest index adjacent to h_1. Then $(H\backslash h_2) \cup \{x, q_1, \ldots, q_r\}$ induces an even wheel with center x. So at least one node of $H\backslash\{h_1, h_j\}$ has a neighbor in Q.

Next suppose that no node of $\{h_2, \ldots, h_{j-1}\}$ has a neighbor in Q. Let p_s be the node of P with highest index adjacent to q_l. If $j > 3$ then $\{x, h_2, \ldots, h_j, p_s, \ldots, p_k\} \cup Q$ induces an even wheel with center x. So $j = 3$. Let $h_{j'}$ be the node of $\{h_{j+1}, \ldots, h_n\}$ with lowest index adjacent to a node of Q. By definition of Q and Lemma 3.2, j' is even. Let q_r be the node of Q with lowest index adjacent to $h_{j'}$. If $j' > 4$ then $\{x, h_j, \ldots, h_{j'}, q_1, \ldots, q_r\}$ induces an even wheel with center x. So $j' = 4$. If $N(h_1) \cap Q = \emptyset$ then $\{x, h_1, h_2, h_3, p_s, \ldots, p_k\} \cup Q$ induces an even wheel with center h_2. So $N(h_1) \cap Q \neq \emptyset$. In fact, by Lemma 3.2, $N(h_1) \cap (Q\backslash q_1) \neq \emptyset$. Suppose $N(h_4) \cap Q \neq q_1$. Let R be a shortest path from h_4 to h_1 in the graph induced by $(Q\backslash q_1) \cup \{h_1, h_4\}$. Then, $\{x, h_1, \ldots, h_4\} \cup R$ induces an even wheel with center x. So $N(h_4) \cap Q = q_1$. Suppose $N(q_l) \cap P \neq p_1$ or $i > 4$. Then $\{x, h_2, h_3, h_4, p_s, \ldots, p_k\} \cup Q$ induces an even wheel with center h_3. So $N(q_l) \cap P = p_1$ and $i = 4$. Let R be a shortest path from p_1 to h_1 in the graph induced by $Q \cup \{p_1, h_1\}$. Then, $P \cup R \cup \{h_1, h_4, x\}$ induces a $3PC(p_1, h_1)$. Therefore, at least one node of $\{h_2, \ldots, h_{j-1}\}$ has a neighbor in Q.

Finally, suppose that no node of $\{h_{j+1}, \ldots, h_n\}$ has a neighbor in Q. Let $h_{j'}$ be a node of $h_2, \ldots, h_{j-1}$ such that $N(h_{j'}) \cap Q \neq \emptyset$ and the path from $h_{j'}$ to h_i in the graph induced by $P \cup Q \cup \{h_i, h_{j'}\}$ is minimized. By definition of Q and Lemma 3.2, j' is even. Suppose $N(h_1) \cap Q \neq \emptyset$. Let R be a shortest path from h_j to h_1 in the graph induced by $Q \cup \{h_1, h_j\}$. Then, $(H\backslash\{h_2, \ldots, h_{j-1}\}) \cup R \cup x$ induces an even wheel with center x. So $N(h_1) \cap Q = \emptyset$. Suppose $N(q_l) \cap P \neq p_k$. Let R be a shortest path from h_i to $h_{j'}$ in the graph induced by $P \cup Q \cup \{h_i, h_{j'}\}$. Note that by definition of Q and $h_{j'}$ and by Lemma 3.2, no node of $\{h_2, \ldots, h_{j'-1}\}$ has a neighbor in R. Then $(H\backslash\{h_{j'+1}, \ldots, h_{i-1}\}) \cup R \cup x$ induces an even wheel with center x. So $N(q_l) \cap P = p_k$. But then $(H\backslash\{h_2, \ldots, h_{j-1}\}) \cup P \cup Q$ induces a $3PC(p_k, h_i)$. This completes the proof of Claim 1. □

By Claim 1 at least two nodes, say $h_{j'}$ and $h_{j''}$, of $H\backslash\{h_1, h_j\}$ have a neighbor in Q. Note that by definition of Q and Lemma 3.2, j' and j'' are both even. W.l.o.g. $j' < j < j''$. Let $R = r_1, \ldots, r_t$ be a shortest path in the graph induced by Q where $N(h_{j'}) \cap R = r_1$ and $N(h_{j''}) \cap R = r_t$. W.l.o.g. and by Lemma 3.2 no other node from $H\backslash\{h_1, h_j\}$ has a neighbor in R.

If $N(h_1) \cap R = \emptyset$, then $(H\backslash\{h_{j'+1}, \ldots, h_{j''-1}\}) \cup R \cup x$ induces an even wheel with center x. So $N(h_1) \cap R \neq \emptyset$. Suppose $j' \neq 2$. Let R' be a shortest path from h_1 to $h_{j'}$ in the graph induced by $R \cup \{h_1, h_{j'}\}$. Then $\{x, h_1, \ldots, h'_j\} \cup R'$ induces an even wheel with center x. Therefore $j' = 2$.

Suppose that $N(h_1) \cap R = r_1$. Then by Lemma 3.2, $N(r_1) \cap H = \{h_1, h_2\}$. If $r_t = q_1$, then by Lemma 3.2, $N(r_t) \cap H = \{h_j, h_{j+1}\}$, and hence $H \cup R$ induces a $3PC(h_1h_2r_1, h_{j+1}h_jr_t)$. So $r_t \neq q_1$, and hence $N(r_t) \cap H = \{h_{j''}\}$. Therefore, $H \cup R$ induces a $3PC(h_1h_2r_1, h_{j''})$. Let R' be a shortest path from q_1 to a node of R in the graph induced by Q. Since $|R \cup R'| < |P \cup Q|$, the choice of P and Q is contradicted.

So $N(h_1) \cap (R\backslash r_1) \neq \emptyset$. Let r_s be the node of R with highest index adjacent to h_1. If h_j has no neighbor in $r_s, \ldots, r_t$, then $\{x, h_1, \ldots, h_{j''}, r_s, \ldots, r_t\}$ induces an even wheel with center x. So h_j does have a neighbor in $r_s, \ldots, r_t$, i.e. $r_t = q_1$. By Lemma 3.2, $N(r_t) \cap H = \{h_j, h_{j''}\}$, where $j'' = j + 1$. Note that $i \geqslant j + 1$ and $r_s \neq q_l$. But then $(H\backslash\{h_2, \ldots, h_j\}) \cup P \cup \{r_s, \ldots, r_t\}$ induces a $3PC(h_1, h_i)$. □

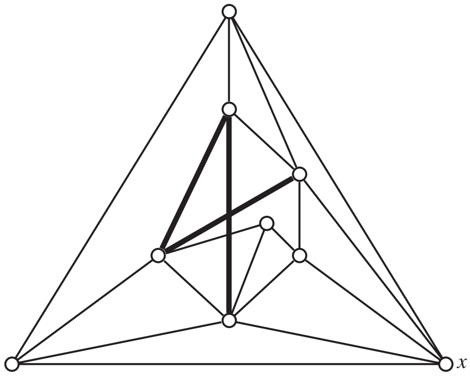

Fig. 2. An odd-signable graph for which Lemma 3.4 does not work.

Note that the above lemma does not work if we allow 4-holes. Consider the odd-signable graph in Fig. 2 (one can see that this graph is odd-signable by assigning 0 to the three bold edges and 1 to all the other edges). Let H be the 5-hole induced by the neighborhood of node x. Then every node of H has a neighbor in the unique connected component obtained by removing $N(x) \cup x$.

Let $\mathscr{F}$ be a class of graphs. We say that a graph G is *$\mathscr{F}$-free* if G does not contain (as an induced subgraph) any of the graphs from $\mathscr{F}$.

A class $\mathscr{F}$ of graphs satisfies *property* (*) *w.r.t. a graph* G if the following holds: for every node x of G such that $G \backslash N[x] \neq \emptyset$, and for every connected component C of $G \backslash N[x]$, if $F \in \mathscr{F}$ is contained in $G[N(x)]$, then there exists a node of F that has no neighbor in C.

The following theorem is proved in [13]. For completeness we include its proof here.

Theorem 3.5 (*Maffray et al. [13]*). *Let $\mathscr{F}$ be a class of graphs such that for every $F \in \mathscr{F}$, no node of F is adjacent to all the other nodes of F. If $\mathscr{F}$ satisfies property* (*) *w.r.t. a graph G, then G has a node whose neighborhood is $\mathscr{F}$-free.*

Proof. Let $\mathscr{F}$ be a class of graphs such that for every $F \in \mathscr{F}$, no node of F is adjacent to all the other nodes of F. Assume that $\mathscr{F}$ satisfies property (*) w.r.t. G, and suppose that for every $x \in V(G)$, $G[N(x)]$ is not $\mathscr{F}$-free. Then G is not a clique (since every graph of $\mathscr{F}$ contains nonadjacent nodes) and hence it contains a node x that is not adjacent to all other nodes of G. Let $C_1, \ldots, C_k$ be the connected components of $G \backslash N[x]$, with $|C_1| \geqslant \cdots \geqslant |C_k|$. Choose x so that for every $y \in V(G)$ the following holds: if $C_1^y, \ldots, C_l^y$ are the connected components of $G \backslash N[y]$ with $|C_1^y| \geqslant \cdots \geqslant |C_l^y|$, then

- $|C_1| > |C_1^y|$, or
- $|C_1| = |C_1^y|$ and $|C_2| > |C_2^y|$, or
- ...
- $|C_1| = |C_1^y|, \ldots, |C_{k-1}| = |C_{k-1}^y|$ and $|C_k| > |C_k^y|$, or
- for $i = 1, \ldots, k$, $|C_i| = |C_i^y|$ and $k = l$.

Let $N = N(x)$ and $C = C_1 \cup \cdots \cup C_k$. For $i = 1, \ldots, k$, let N_i be the set of nodes of N that have a neighbor in C_i.

Claim 1. *$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k$ and for every $i = 1, \ldots, k-1$, every node of $(N \backslash N_i) \cup (C_{i+1} \cup \cdots \cup C_k)$ is adjacent to every node of N_i.*

Proof of Claim 1. We argue by induction. First we show that every node of $(N \backslash N_1) \cup (C_2 \cup \cdots \cup C_k)$ is adjacent to every node of N_1. Assume not and let $y \in (N \backslash N_1) \cup (C_2 \cup \cdots \cup C_k)$ be such that it is not adjacent to $z \in N_1$. Clearly y has no neighbor in C_1, but z does. So $G \backslash N[y]$ contains a connected component that contains $C_1 \cup z$, contradicting the choice of x.

Now let $i > 1$ and assume that $N_1 \subseteq \cdots \subseteq N_{i-1}$ and every node of $(N \backslash N_{i-1}) \cup (C_i \cup \cdots \cup C_k)$ is adjacent to every node of N_{i-1}. Since every node of C_i is adjacent to every node of N_{i-1}, it follows that $N_{i-1} \subseteq N_i$. Suppose that there exists a node $y \in (N\backslash N_i) \cup (C_{i+1} \cup \cdots \cup C_k)$ that is not adjacent to a node $z \in N_i$. Then $z \in N_i\backslash N_{i-1}$ and z has a neighbor in C_i. Also y is adjacent to all nodes in N_{i-1} and no node of $C_1 \cup \cdots \cup C_i$. So there exist connected components of $G\backslash N[y]$, $C_1^y, \ldots, C_l^y$ such that $C_1 = C_1^y, \ldots, C_{i-1} = C_{i-1}^y$ and $C_i \cup z$ is contained in C_i^y. This contradicts the choice of x. This completes the proof of Claim 1. $\square$

Since $G[N]$ is not $\mathscr{F}$-free, it contains $F \in \mathscr{F}$. By property (*), a node y of F has no neighbor in C_k. By Claim 1, y is adjacent to every node of N_k, and no node of $N\backslash N_k$ has a neighbor in C. So (since every node of F has a non-neighbor in F) F must contain another node $z \in N\backslash N_k$, nonadjacent to y. But then $C_1, \ldots, C_k$ are connected components of $G\backslash N[y]$ and z is contained in $(G\backslash N[y])\backslash C$, so y contradicts the choice of x.

Proof of Theorem 1.1. Let G be a 4-hole-free odd-signable graph. Let $\mathscr{F}$ be the set of all holes. By Lemma 3.4, $\mathscr{F}$ satisfies property (*) w.r.t. G. So by Theorem 3.5, G has a node whose neighborhood is $\mathscr{F}$-free, i.e. triangulated. $\square$

4. Final remarks

In a graph G, for any node x, let $C_1, \ldots, C_k$ be the connected components of $G\backslash N[x]$, with $|C_1| \geqslant \cdots \geqslant |C_k|$, and let the numerical vector $(|C_1|, \ldots, |C_k|)$ be associated with x. The nodes of G can thus be ordered according to the lexicographic ordering of the numerical vectors associated with them. Say that a node x is *lex-maximal* if the associated numerical vector is lexicographically maximal over all nodes of G. Theorem 3.5 actually shows that for a lex-maximal node x, $N(x)$ is $\mathscr{F}$-free. This implies the following.

Theorem 4.1. *Let G be a* 4-*hole-free odd-signable graph, and let x be a lex-maximal node of G. Then the neighborhood of x is triangulated.*

Possibly a more efficient algorithm for listing all maximal cliques can be constructed by searching for a lex-maximal node.

Lemma 3.4 also proves the following decomposition theorem. (H, x) is a *universal wheel* if x is adjacent to all the nodes of H. A node set S is a *star cutset* of a connected graph G if for some $x \in S$, $S \subseteq N[x]$ and $G\backslash S$ is disconnected.

Theorem 4.2. *Let G be a* 4-*hole-free odd-signable graph. If G contains a universal wheel, then G has a star cutset.*

Proof. Let (H, x) be a universal wheel of G. If $G = N[x]$, then for any two nonadjacent nodes a and b of H, $N[x]\backslash\{a, b\}$ is a star cutset of G. So assume $G\backslash N[x]$ contains a connected component C_1. By Lemma 3.4, a node $a \in H$ has no neighbor in C_1. But then $N[x]\backslash a$ is a star cutset of G that separates a from C_1. $\square$

In [7] universal wheels in 4-hole-free odd-signable graphs are decomposed with triple star cutsets, i.e. node cutsets S such that for some triangle $\{x_1, x_2, x_3\} \subseteq S$, $S \subseteq N(x_1) \cup N(x_2) \cup N(x_3)$.

References

[1] L. Addario-Berry, M. Chudnovsky, F. Havet, B. Reed, P. Seymour, Bisimplicial vertices in even-hole-free graphs, preprint, 2006.
[2] N. Chiba, T. Nishizeki, Arboricity and subgraph listing algorithms, SIAM J. Comput. 14 (1985) 210–223.
[3] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, K. Vušković, Recognizing Berge graphs, Combinatorica 25 (2005) 143–187.
[4] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Ann. Math. 164 (2006) 51–229.
[5] M. Chudnovsky, P. Seymour, Excluding induced subgraphs, preprint, 2006.
[6] M. Conforti, G. Cornuéjols, A. Kapoor, K. Vušković, Even and odd holes in cap-free graphs, J. Graph Theory 30 (1999) 289–308.
[7] M. Conforti, G. Cornuéjols, A. Kapoor, K. Vušković, Even-hole-free graphs, Part I: decomposition theorem, J. Graph Theory 39 (2002) 6–49.
[8] M. Conforti, G. Cornuéjols, A. Kapoor, K. Vušković, Even-hole-free graphs Part II: recognition algorithm, J. Graph Theory 40 (2002) 238–266.
[9] D. Coppersmith, S. Winograd, Matrix multiplication via arithmetic progression, J. Symbolic Comput. 9 (1990) 251–280.
[10] M. Farber, On diameters and radii of bridged graphs, Discrete Math. 73 (1989) 249–260.
[11] D.R. Fulkerson, O.A. Gross, Incidence matrices and interval graphs, Pacific J. Math. 15 (1965) 835–855.
[12] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, second ed., Annals of Discrete Mathematics, vol. 57, Elsevier, Amsterdam, 2004.

[13] F. Maffray, N. Trotignon, K. Vušković, Algorithms for square-$3PC(\cdot, \cdot)$-free Berge graphs, preprint, 2005, submitted for publication.
[14] K. Makino, T. Uno, New algorithm for enumerating all maximal cliques, in: T. Hagerup, J. Katajainen (Eds.), Algorithm Theory—SWAT 2004, Lecture Notes in Computer Science, vol. 3111, 2004, pp. 260–272.
[15] I. Parfenoff, F. Roussel, I. Rusu, Triangulated neighborhoods in C_4-free Berge graphs, in: Proceedings of WG'99, Lecture Notes in Computer Science, vol. 1665, 1999, pp. 402–412.
[16] D.J. Rose, R.E. Tarjan, G.S. Leuker, Algorithmic aspects of vertex elimination on graphs, SIAM J. Comput. 5 (1976) 266–283.
[17] J. Spinrad, Efficient Graph Representations, Field Institute Monographs, vol. 19, American Mathematical Society, Providence, RI, 2003.
[18] K. Truemper, Alpha-balanced graphs and matrices and GF(3)-representability of matroids, J. Combin. Theory B 32 (1982) 112–139.
[19] S. Tsukiyama, M. Ide, H. Ariyoshi, I. Shirakawa, A new algorithm for generating all the maximal independent sets, SIAM J. Comput. 6 (1977) 505–517.