The Chromatic Index of Proper Circular Arc Graphs of Odd Maximum Degree which are Chordal

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Abstract

Although the edge-coloring problem is NP-hard for graphs in general, the problem is partially solved for proper interval graphs, a subclass of proper circular arc graphs, by a technique called pullback. Furthermore, Figueiredo, Meidanis, and Mello conjectured in the late 1990s that all chordal graphs of odd maximum degree Δ have chromatic index equal to Δ. Using a new technique called multi-pullback, we show that this conjecture holds for chordal $\cap$ proper circular arc graphs of odd Δ.

Keywords: Pullback, circular arc, chromatic index, edge-coloring, chordal

1 Introduction

Circular arc graphs are the intersection graphs of a finite set of arcs on a circle. If no arc properly contains another, the graph is said to be a a proper circular arc graph. If all the arcs have the same length, the graph is said to be a unit circular arc graph. Although the class of the circular arc graphs is well studied, very little is known about deciding the chromatic index of these graphs, except for the subclass consisting of the n-vertex proper circular arc graphs of odd maximum degree Δ which have $n \not\equiv 1, \Delta \pmod{(\Delta+1)}$ and a maximal clique of size two, or which have $n \equiv 0 \pmod{(\Delta+1)}$ [1].

Circular arc graphs are a superclass of interval graphs. An important difference between these two classes is that interval graphs have a linear number of maximal cliques (in the number of vertices), while circular arc graphs may have an exponential number of maximal cliques. This may suggest why some problems are more difficult for circular arc graphs than for interval graphs. For instance, vertex-coloring is polynomial for interval graphs, but NP-hard for circular arc graphs [4].

The NP-hard edge-coloring problem asks the minimum amount of colors needed to color the edges of a graph in such a way that no two adjacent edges receive the same color. This amount is called the chromatic index of G, denoted $\chi'(G)$. By definition, $\chi'(G) \geq \Delta(G)$ for any graph G. The celebrated Vizing's Theorem brings that $\chi'(G) \leq \Delta(G) + 1$ [8]. Therefore, we say that a graph is *Class 1* if $\chi'(G) = \Delta(G)$, or *Class 2* if $\chi'(G) = \Delta(G) + 1$. For instance, a complete graph K_n is *Class 1* if n is even, and *Class 2* otherwise.

We solve the edge-coloring problem in the class of proper circular arc $\cap$ chordal (PCAC) graphs of odd maximum degree, that is, we prove that all of these graphs are *Class 1*, and our proof yields a polynomial-time

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exact edge-coloring algorithm for these graphs. It is important to remark that even for proper interval graphs (often referred to as *indifference* graphs in the literature), an important subclass of proper circular arcs, the problem is solved only for graphs with odd maximum degree, by a technique called *pullback* [2]. Later, this technique was also used to solve the edge-coloring problem for all dually chordal graphs (a superclass of interval graphs) of odd maximum degree [3].

To solve the problem for the PCAC graphs of odd maximum degree, we design a new technique called *multi-pullback*, which we suspect that can be used for other classes, including the indifference graphs with even maximum degree.

This paper is organized as follows: the remaining of this section is dedicated to some preliminary definitions; Section 2 discusses the pullback functions introduced in [2] and presents our multi-pullback functions; then, Section 3 presents our results on PCAC graphs using the multi-pullback functions introduced in Section 2.

Preliminary definitions

In this paper, graph-theoretical definitions follow their usual meanings in the literature. In particular, $G = (V(G), E(G))$ is a graph, $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. An edge uv is said to be *incident* to the vertices u and v, and the vertices u and v are said to be *neighbors*. The *degree* of a vertex u, denoted $d_G(u)$, is the number of edges that are incident to the vertex u. The *maximum degree* of G is $\Delta(G) := \max\{d_G(u) : u \in V(G)\}$. The *open neighborhood* of a vertex is the set $N_G(u) := \{v : uv \in E(G)\}$. The *closed neighborhood* of a vertex is the set $N_G[u] := N_G(u) \cup \{u\}$. We say that a graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Let $U \subset V(G)$. The subgraph of G *induced* by U is defined by $G[U] := (U, \{uv \in E(G) : u, v \in U\})$. Let $F \subset E(G)$. The subgraph of G *induced* by F is defined by $G[F] := (\{u : uv \in F \text{ for some } v \in V(G),\ F)$. The *core* of a graph is the subgraph induced by its vertices of maximum degree. The *semi-core* of a graph is the subgraph induced by the vertices of maximum degree and their neighbors. A *k-edge-coloring* of G is a proper edge-coloring of G with k colors, that is, an assignment of colors to the edges of a graph in such a way that no two adjacent edges receive the same color and that at most k colors are used. A set $U \subset V(G)$ is said to be a *clique* if it induces a complete graph in G. A clique is said to be *maximal* if it is not properly contained in any other clique. A *simplicial vertex* in G is a vertex that belongs to only one maximal clique of G.

2 Pullback and multi-pullback functions

A function $f : V(G) \to V(G')$ is said to be a *pullback* if it is a homomorphism (i.e. for all $uv \in E(G)$ we have $f(u)f(v) \in E(G')$), and if f is injective when restricted to $N_G[u]$ for all $u \in V(G)$.

Lemma 2.1 ([2,3]) *If f is a pullback from G to G' and λ' is an edge-coloring of G', then the function $\lambda(uv) := \lambda'(f(x)f(y))$ is an edge-coloring of G.*

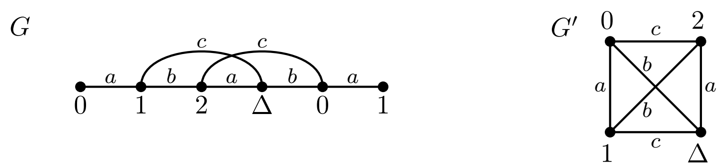

Figure 1. Example of a pullback from an indifference graph G to $G' := K_4$

Definition 2.2 Let $G = (V, E)$ be a graph with $E \neq \emptyset$ and let $\{E_1, \ldots, E_t\}$ be a partition of E. A *multi-pullback* F from G to a collection of t graphs $\{G'_1, \ldots, G'_t\}$ is a collection of t functions $\{f_1, \ldots, f_t\}$ such that:

(i) f_i is a pullback from $G[E_i]$ to G'_i;

(ii) there is some positive integer k and some collection of k-edge-colorings $\lambda'_1, \ldots, \lambda'_t$ of $G'_1, \ldots, G'_t$, respectively, such that the edge-colorings obtained from $\lambda'_1, \ldots, \lambda'_t$ and the pullbacks $f_1, \ldots, f_t$ do not create any color conflict on the edges of G, that is, the function defined by

$$\lambda(uv) := \lambda'_i(f_i(u)f_i(v)), \quad \text{being } E_i \textit{ the} \text{ set of the partition to which } uv \text{ belongs,}$$

is a proper k-edge-coloring of G.

In Definition 2.2, observe that disjointness is assumed only among the sets of the partition $\{E_1, \ldots, E_t\}$, but not among the domains of the functions in F, which are sets of vertices, not edges. This means that a single vertex u can be mapped to a vertex v of G'_i by a pullback f_i and to a different vertex w of G'_j by a pullback f_j, depending on which *role* we want u to assume in order to color each edge incident to u. The Figure 2 shows an example of a collection of functions $\{f_1, f_2, f_3\}$ which can be verified to be a multi-pullback from a PCAC graph G with $\Delta = 5$ to the K_6, under the 5-edge-colorings $\lambda'_1 = \lambda'_2 = \lambda'_3 =: \lambda'$ of the K_6 defined by

$$\lambda'(uv) = \begin{cases} (u+v) \bmod \Delta\,, & \text{if neither } u \text{ nor } v \text{ is } \Delta; \\ (2v) \bmod \Delta\,, & \text{if } u = \Delta. \end{cases} \tag{1}$$

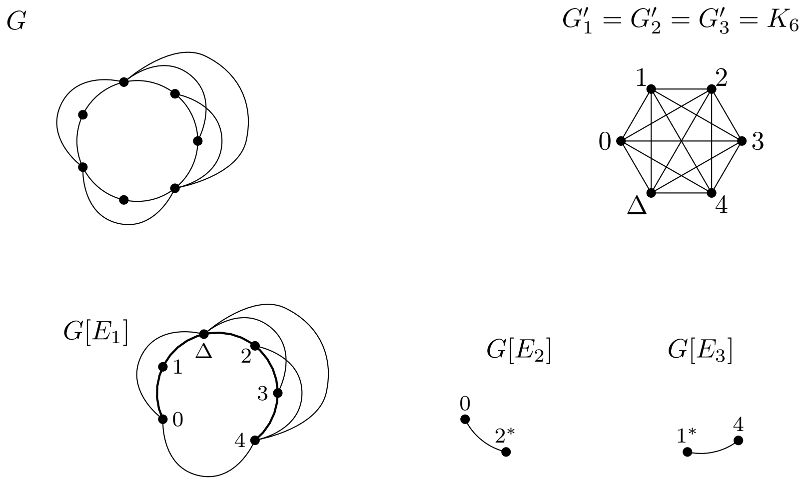

Figure 2. Example of a multi-pullback from G to the K_6 under the 5-edge-coloring defined in (1). Observe that the vertex marked with an asterisk is mapped to two distinct vertices by f_2 and by f_3, with no color conflict being created.

3 The result

Proper circular arc graphs have the consecutive 1's property [7], i.e. there is a circular order for the vertices in such a way that for every edge $\overrightarrow{uv}$ under the clockwise orientation of the edges along this order, all the vertices clockwise between u and v induce a complete graph in the original undirected graph. This order is called a proper circular arc order.

Lemma 3.1 *Let G be a PCAC graph of odd maximum degree. If G has a universal vertex, or if the semi-core of G is an indifference graph, then G is* Class 1.

Proof Observe first that if G has a universal vertex, then G is a subgraph of the $K_{\Delta(G)+1}$ and hence *Class 1*. On the other hand, if the semi-core of G is an indifference graph, then G is also *Class 1* because the chromatic index of a graph is equal to the chromatic index of its semi-core [5], and because all indifference graphs of odd maximum degree are *Class 1* [2]. □

Lemma 3.2 below provides a full characterization of the structure of proper circular arc ∩ chordal graphs which do not satisfy Lemma 3.1.

Lemma 3.2 *If G is a PCAC graph of odd maximum degree with no universal vertex such that the semi-core S of G is not an indifference graph, then $S = G$ and there is a 6-partition $\{A, AB, B, BC, C, AC\}$ of $V(G)$ which splits any proper circular arc order of G into six contiguous subsequences of the order in a manner that, being the cardinality of each set in the partition denoted by the corresponding lowercase letters:*

(i) *the graph G has exactly four maximal cliques, which can be given by $X_A := \{AB \cup A \cup AC\}$, $X_B := \{BC \cup B \cup AB\}$, $X_C := \{AC \cup C \cup BC\}$, and $Y = \{AB \cup AC \cup BC\}$;*

(ii) *X_A is assumed without loss of generality to be of maximum size among the three cliques which appear contiguously in the proper circular arc order (that is, all the vertices in each of these cliques appear consecutively in the order), which are the cliques X_A, X_B, and X_C;*

(iii) *all the vertices in AB and in AC have degree $\Delta(G)$ in G;*

(iv) $\Delta(G) = a + b + ab + bc + ac - 1 = a + c + ab + bc + ac - 1$;

(v) $a \geq b = c$;

Proof Let σ be a proper circular arc order of G and let $(X_0, X_1, \cdots, X_{t-1})$ be the maximal cliques that appear contiguously in σ. We must have $t \geq 3$, otherwise it can be straightforwardly shown that G would be an indifference graph.

We claim that there is no X_i such that $X_i \subset X_{(i-1) \bmod t} \cup X_{(i+1) \bmod t}$. If this claim holds, an induced cycle of size t is easily obtained by choosing one vertex from each $X_i \cap X_{(i+1) \bmod t}$. Because G is chordal and it is not an indifference graph, we have $t = 3$. These three maximal cliques of G that appear contiguously in σ are X_A, X_B, and X_C, respectively.

Since G is not an indifference graph, we have that the intersection of two consecutive cliques in σ is not empty (otherwise in any circular arc model of G there would be a point on the circumference which would be uncovered by any arc). We define the sets $AB := X_A \cap X_B$, $BC := X_B \cap X_C$, and $AC := X_A \cap X_C$, and also $A := X_A \setminus (AB \cup AC)$, $B := X_B \setminus (AB \cup BC)$, and $C := X_C \setminus (AC \cup BC)$. As all the vertices in $AB \cup BC \cup AC$ are neighbors of each other, there is a fourth maximal clique $Y := AB \cup BC \cup AC$ that does not appear contiguously in σ.

Up to this point, we have proven that if the claim holds then there are at least three maximal cliques (X_A, X_B, and X_C) which appear contiguously in σ, as well as the fourth clique Y. We have also proven that the sets AB, BC, and AC are not empty. We can further demonstrate that the sets A, B, and C are non-empty, which is equivalent to prove that each of the cliques X_A, X_B, and X_C has a simplicial vertex. If $A = \emptyset$, then every vertex of BC is universal (see Figure 3), contradicting the hypothesis. The non-emptiness of B and C follows analogously.

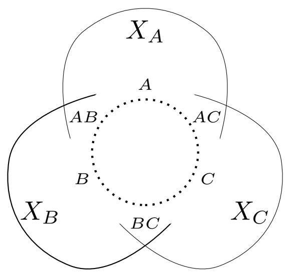

Figure 3. Structure of a PCAC graph according to Lemma 3.2.

Now we shall prove the claim and that X_A, X_B, and X_C are the only maximal cliques contiguously in σ. Assume for the sake of contradiction that there is a fourth maximal clique X_D contiguously in σ. Since the intersections AB, BC, and AC are all non-empty, the clique X_D must be contained in the union of two cliques from $\{X_A, X_B, X_C\}$. Without loss of generality, $X_D \subset X_A \cup X_B$. By the same arguments presented above, the four sets $(X_D \cap X_A) \setminus (X_B \cup X_C)$, $(X_D \cap X_B) \setminus (X_A \cup X_C)$, $(X_B \cap X_C) \setminus X_D$, and $(X_C \cap X_A) \setminus X_D$ are all non-empty. Ergo, by choosing four vertices, one from each of these sets, we obtain an induced cycle of size four, contradicting the fact that G is chordal. Hence, we have proven that X_D cannot exist and also that the claim holds. Furthermore, since all the vertices in A, B, and C are simplicial, the only maximal clique which can be formed not contiguously in σ is the clique Y (recall Figure 3).

Assuming without loss of generality that X_A is of maximum size among X_A, X_B, and X_C, it remains to demonstrate (iii)–(v). Clearly, the vertices of maximum degree in G are in $AB \cup BC \cup AC$. We shall demonstrate that either AB and AC, or all the sets from $\{AB, AC, BC\}$ have vertices of maximum degree (this proves (iii)). If only one set I from $\{AB, AC, BC\}$ has vertices of maximum degree in G, then surely $I \neq BC$, because of the assumption on the cardinality of X_A. If $I = AB$, then the semi-core of G is an indifference graph, because the order B, BC, AB, AC, A is an indifference order[7]. The case $I = AC$ follows analogously. Remark that this also proves that the semi-core of G equals G.

Notice that vertices which belong to the same set from $\{A, AB, B, BC, C, AC\}$ have the same closed neighborhood and hence the same degree. Let u be a vertex in AB, v a vertex in AC, and w a vertex in BC. We

[7] An indifference order of an indifference graph is a linear order of the vertices so that vertices belonging to the same maximal clique appear consecutively in this order [6].

know that $\Delta(G) = d_G(u) = d_G(v) \geq d_G(w)$ and also that:

$$\begin{aligned} d_G(u) &= bc + b + ab + a + ac - 1\,; \\ d_G(v) &= ab + a + ac + c + bc - 1\,; \\ d_G(w) &= ab + b + bc + c + ac - 1\,. \end{aligned}$$

From these equations, we have (iv) and also that $b = c$ and $a \geq b$, completing the proof of (v). □

Theorem 3.3 *Every proper circular arc ∩ chordal graph with odd maximum degree is Class 1.*

Proof In view of Lemma 3.1, let G be a PCAC graph of odd maximum degree with no universal vertex such that the semi-core of G is not an indifference graph. Let also $\{A, AB, B, BC, C, AC\}$ be a partition of $V(G)$ as in Lemma 3.2 (recall Figure 3). Let $\{E_1, E_2, E_3, E_4\}$ be the partition of $E(G)$ defined by:

$$\begin{aligned} E_1 &\coloneqq E(G[A \cup AB \cup B \cup BC \cup AC])\,; \\ E_2 &\coloneqq \{uv : u \in BC \text{ and } v \in C\}\,; \\ E_3 &\coloneqq \{uv : u \in AC \text{ and } v \in C\}\,; \\ E_4 &\coloneqq E(G[C])\,. \end{aligned}$$

Let $V(K_{\Delta(G)+1}) = \{0, \ldots, \Delta(G)\}$ and $V(K_c) = \{0, \ldots, c-1\}$. We shall construct a multi-pullback $\{f_1, f_2, f_3, f_4\}$ with $f_i \colon V_i \to G'_i$, for all $i \in \{1, \ldots, 4\}$, being

$$\begin{aligned} V_1 &\coloneqq A \cup AB \cup B \cup BC \cup AC\,, \\ V_2 &\coloneqq BC \cup C\,, \\ V_3 &\coloneqq AC \cup C\,, \\ V_4 &\coloneqq C\,, \end{aligned}$$

and being $G'_1 \coloneqq G'_2 \coloneqq G'_3 \coloneqq K_{\Delta(G)+1}$ and $G'_4 = K_c$, under the edge-colorings $\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4$ defined by $\lambda'_1 \coloneqq \lambda'_2 \coloneqq \lambda'_3 \coloneqq \lambda'$, wherein λ' is the $\Delta(G)$-edge-coloring of the $K_{\Delta(G)+1}$ defined in (1), and λ'_4 is the $\Delta(G)$-edge-coloring of the K_c defined by

$$(u, v) = (2ac + a + c + bc + u + v) \bmod \Delta(G)\,.$$

Remark that λ' is an optimal edge-coloring of the $K_{\Delta(G)+1}$ (which is *Class 1* since $\Delta(G)$ is odd) and that λ_4 is surely not optimal, since $c < \Delta(G) - 2$.

Remark by Lemma 3.2 that $|V_1| = \Delta(G) + 1$. In order to define f_1, take any bijective labeling function satisfying:

$$\begin{aligned} f_1(AC) &= \{0, \ldots, ac - 1\}\,; \\ f_1(A) &= \{ac, \ldots, ac + a - 1\}\,; \\ f_1(B) &= \{ac + a, \ldots, ac + a + b - 1\}\,; \\ f_1(BC) &= \{ac + a + b, \ldots, ac + a + b + bc - 1\}\,; \\ f_1(AB) &= \{ac + a + b + bc, \ldots, ac + a + b + bc + ab - 1\}\,. \end{aligned}$$

Here, we use $f_1(Z)$ to denote $\bigcup_{z \in Z}\{f_1(z)\}$. Notice that we have used $\Delta(G)+1$ distinct labels, from 0 to $\Delta(G)$, and it is easy to realize that this labeling is a *pullback* from $G[E_1]$ to the G'_1.

It remains to color the edges incident to the vertices of C, that is, it remains to define $f_2, \ldots, f_4$. Remark that $G[E_2 \cup E_3]$ is a bipartite graph, with parts C and $BC \cup AC$, and $G[E_4]$ is a complete graph. Figure 4 represents the sets BC, C, and AC.

Recall that $G[E_2]$ is the bipartite graph induced by the edges between AC and C, and notice that the edges incident to vertices in AC are not incident to vertices in B. This is why we can define f_2 by assigning to the vertices of AC the same labels which they have been assigned by f_1, and to the vertices of B the same labels assigned to the vertices of C by f_1, in the manner that we clarify in the sequel. As $b = c$, there will be enough labels for all the vertices of C.

Analogously, the graph $G[E_3]$ is the bipartite graph induced by the edges between BC and C. Notice that vertices in BC are not neighbors of vertices in A, therefore, the labels assigned by f_1 to the vertices in A can

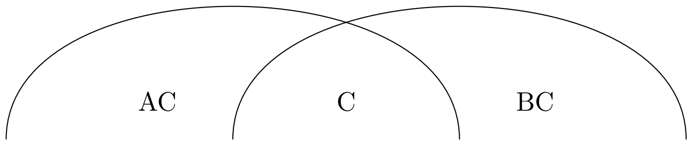

Figure 4. The sets BC, C, and AC

be reused by f_3 to the vertices in C (in the manner that we clarify in the sequel), if the vertices in BC are assigned by f_3 the same labels which they have been assigned by f_1. Recall that $a \geq c$, so there will be enough labels.

To complete the proof, it remains only to define which are the three labels assigned to each vertex in C by f_2, f_3, and f_4, and to show that the edge-coloring obtained through these pullbacks do not create color conflicts in G. Let $C = \{u_0, \ldots, u_{c-1}\}$. We define for each $u_i \in C$ the triplet $(f_2(u_i), f_3(u_i), f_4(u_i)) \coloneqq (ac+a+i, ac+i, i)$. Let λ be the $\Delta(G)$-edge-coloring of G as in Definition 2.2. We show that λ is a proper edge-coloring, for which it suffices to show that all the colors of the edges incident to the same vertex u_i in C are different.

The colors of the edges incident to u_i can be verified to be as follows (all the colors listed below are $\bmod \Delta(G)$, but this information is omitted for a clear description):

- the colors of the edges of $G[E_2]$ that are incident to u_i are the ac colors from the set

$$\{ac + a + i, \ldots, 2ac + a + i - 1\}\,;$$

- the colors of the edges of $G[E_3]$ that are incident to u_i are the bc colors from the set

$$\{2ac + a + b + i, \ldots, 2ac + a + b + bc + i - 1\}\,;$$

- the colors of the edges of $G[E_4]$ that are incident to u_i are the $c-1$ colors from the set

$$\{2ac + a + b + bc + i, \ldots, 2ac + a + 2b + bc + i\} \setminus \{2ac + a + b + bc + 2i\}\,;$$

Notice that, at the edges incident to u_i, the b colors between $(2ac + a + i) \bmod \Delta(G)$ and $(2ac + a + b + i - 1) \bmod \Delta(G)$ are not used, as well as the color $(2ac + a + b + bc + 2i) \bmod \Delta(G)$. As $ac + b + bc + c \leq \Delta(G) = ac + bc + ab + a + c - 1$, there is no color conflict at u_i.

Since we have shown that there is no color conflict at any vertex $u_i \in C$, we conclude that G is *Class 1*. $\square$

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