Convergence Time to Nash Equilibrium in Selfish Bin Packing [⋆]

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Abstract

We consider a game-theoretic *bin packing problem* and we study the convergence time to a *Nash equilibrium*. We show that, if the *best-response* strategy is used, then the number of steps needed to reach Nash equilibrium is $O(mw_{\max}^2 + nw_{\max})$ and $O(nkw_{\max})$, where n, m, k and $w_{\max}$ denotes, resp., the number of items, the number of bins, the number of distinct item sizes, and the size of a largest item.

Keywords: analysis of algorithms, selfish bin packing, convergence time.

1 Introduction

In large-scale systems, *e.g.* the Internet, it is difficult or impossible to maintain a central authority who organizes or dictate rules about the actions to be taken on many systems. In face of that, the actions are taken by entities, called *players*, belonging to the system, each with an own goal. Each player chooses his action based on the current state of the system, which in turn is determined

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by the actions of other players. Thus, a player updates his action in response to the actions of others so that a sequence of actions occurs. This sequence may stop at a steady state where no player wishes to update its action, or may continue indefinitely. This steady state is called the *Nash equilibrium*, and is considered the main concept of solution for non-cooperative games, *i.e.*, games where players act in an *independent and selfish* way.

In this work, we are interested in the *bin packing* problem where the existence of a central authority is infeasible. Thus, each item can be seen as a selfish player who wants to act so as to achieve its goal. More specifically, we are interested in the number of steps (*i.e.*, updates of actions) to be made by the system to achieve the Nash equilibrium. In our model, we assume the *elementary stepwise system* (ESS), *i.e.*, at each step only one item updates its action. In *bin packing game*, we have n items and m bins. All bins have the same size C and cost equal to one, each item i has integer size w_i. Let ℓ_j be the load of bin j, *i.e.*, the total size of items assigned to bin j. An item i assigned to a bin j pays the fraction of the load he is using, or w_i/ℓ_j. As i is selfish and, therefore, wants to minimize his cost, it migrates to another bin j' if $w_i + \ell_{j'} \leq C$ and $w_i/(w_i + \ell_{j'}) < w_i/\ell_j$ (*i.e.* $w_i + \ell_{j'} > \ell_j$). That is, i moves from j to j' if it fits in j' and its new cost is smaller. The Nash equilibrium is a feasible packing where no player can reduce its cost moving to another bin.

Related Work: A game theoretic model for bin packing was first proposed by Bilò in [1]. He proved that bin packing game in the ESS model always converges to Nash equilibrium, showing an $O(P^2)$ upper bound in the number of steps, where P is the sum of the sizes of all items. He also proved bounds on the price of anarchy. In [2], Epstein and Kleiman obtained better bounds for the price of anarchy. In [5], Yu and Zhang show that computing a pure Nash equilibrium can be done in polynomial time, although it requires a centralized algorithm. In [4], Miyazawa and Vignatti show logarithms and polynomial bounds for the convergence time in a distributed setting of selfish bin packing. To the present, these four papers are the only ones to address the bin packing problem under a game theoretic perspective. The bin packing problem is also related with the *load balancing* problem. Bilò [1] observed some similarities between these two problems and used a potential function to prove convergence time in a similar way as done for the load balancing problem [3]. Even-dar *et al.* [3] show lower and upper bounds on the number of steps to reach a Nash equilibrium of load balancing problems in ESS in many cases.

Our Results: We present two upper bounds of convergence time to Nash equilibrium in the bin packing game, when using the *best-response* strategy, *i.e.*, when a player moves to a bin with the lowest cost for him. We show that the number of steps needed to reach Nash equilibrium is $O(mw_{\max}^2 + nw_{\max})$ (Theorem 3.7) and $O(nkw_{\max})$ (Theorem 3.8), where k denotes the number of distinct item sizes and $w_{\max}$ the size of a largest item. It is worth noting that our results are the first non-trivial bounds for the problem, and our proof techniques are different from those in [1,3] and from other papers regarding covergence time to Nash equilibrium.

2 Model Description

The model is composed by m bins with capacity C, each one with cost 1, and n items with *sizes* $w_1, \ldots, w_n$. Let $[n]$ be the set of items, and $[m]$ the set of bins. A function $A : [n] \to [m]$ represents a configuration of the game if for any bin j, its *load* ℓ_j is at most C, where $\ell_j = \sum_{i \in [n]: j = A(i)} w_i$. Each item is controlled by a player, who wants to assign his item in a selfish way to a less costly bin. When player i uses bin j, he pays $\frac{w_i}{\ell_j}$. Let i be an item assigned to bin j. As i wants to minimize his cost, he will migrate if he finds a bin $j' \neq j$ such that $w_i + \ell_{j'} \leq C$ and $w_i/(w_i + \ell_{j'}) < w_i/\ell_j$ (*i.e.* $w_i + \ell_{j'} > \ell_j$).

Throughout the text, we use n, m, w_i and ℓ_j as defined above. We also use $w_{\min}$ and $w_{\max}$ to denote, resp., the smallest and the largest size of an item, and $k \leq n$ to denote the number of distinct sizes in the set. Let $w_{\min} = s_1, \ldots, s_k = w_{\max}$ be the k different sizes sorted in increasing order.

To make the notion of time precise, we define a time t as the moment of the game just after the t-th move. Let $W_i^t = \sum_j \max(0, \ell_j - (C - s_{k-i}))$, for each time $t \geq 0$. To simplify the notation, we use W_i in situations regarding one specific move, where the time t is not needed.

3 Bounds on the Convergence Time

In this section, we prove two bounds for the convergence time to Nash equilibrium, given in Theorems 3.7 and 3.8. We first present some definitions and technical results before presenting these two theorems.

Lemma 3.1 *During the game, W_i never decreases and is at most ms_{k-i}. That is $W_i^0 \leq W_i^1 \leq W_i^2 \leq \ldots \leq ms_{k-i}$.*

Lemma 3.1 is easy to prove and its proof is omitted. From Lemma 3.1, the number of steps that increase W_i is at most ms_{k-i}.

We say that a bin j is *light* if $\ell_j \leq C - w_{\max}$, otherwise, it is said to be a *heavy* bin. Thus, there are 4 types of moves: (1) light to light, (2) light to heavy, (3) heavy to light and (4) heavy to heavy. In the next lemmas, we bound the number of moves of each type.

Lemma 3.2 *There are at most $mw_{\max}$ moves of type (2).*

Proof. Consider that the t-th move is of type (2). When an item leaves a light bin, W_0^t is not changed, and when it arrives in the heavy bin, W_0^t increases at least the size of the item. The result follows from Lemma 3.1. □

Lemma 3.3 *There are at most $mw_{\max}$ moves of type (3).*

Proof. Suppose that the item leaves the heavy bin j and goes to the light bin j' in time t. Since items move selfishly, after the move, the load of bin j' is greater than the load of bin j before the move. Therefore, bin j' becomes heavy and the move contribution for W_0^t is at least 1 (even if bin j becomes a light bin). The result follows from Lemma 3.1. □

Lemma 3.4 *There are at most $mw_{\max}^2$ moves of type (4).*

Proof. For each bin we assign *tokens*. A bin can have multiple tokens assigned to it, and the assignment and creation of the tokens is done as follows. Each created token is distinct from each other. A token is *created* in two situations, *(i)* when a light bin becomes heavy, we create the token and assign it to this bin or *(ii)* when an item from a light bin migrates to a heavy bin, we create a token and add it to the heavy bin. Note that in both cases, W_0^t increases by at least 1, and from Lemma 3.1, the total number of created tokens is at most $mw_{\max}$. If an item moves from bin j to bin j', then *all* the tokens assigned to j are reassigned to j'. Therefore, the tokens are always moving to more filled bins. Thus, looking at a specific token, it moves to more filled bins at most $w_{\max}$ times, and therefore there are at most $w_{\max}$ moves associated with that token. As we have at most $mw_{\max}$ tokens and for each token we have at most $w_{\max}$ moves, the result follows. □

Lemma 3.5 presents a result that will be useful in Theorem 3.8 and to bound the number of moves of type (1). Before that, we present some definitions used throughout the text. We define $k+1$ loading intervals, denoted by *regions*, $L_0, \ldots, L_k$, where L_i is the region bounded by load greater than $C - s_{k-i+1}$ and at most $C - s_{k-i}$, for $i \geq 1$. Region L_0 is bounded by load between 0 and $C - s_k$. Let H_i be the region bounded by load greater than $C - s_{k-i}$. Let $L_i \to L_j$ denote a move of an item from a bin whose load is in region L_i to another bin whose load is in region L_j; similarly, we define $L_i \to H_j$, $H_j \to L_i$ and $H_i \to H_j$.

Lemma 3.5 *There are $O(nw_{\max})$ moves of type $L_i \to L_i$.*

Proof. Let $i \in \{0, \dots, k\}$. A bin is said to be *underloaded* if its load is at most $C - s_{k-i}$, otherwise, it is said to be *overloaded*. The *heaviest underloaded bins* are the underloaded bins with the greatest load in an instant of time. Let j^* be a pointer to one of the heaviest underloaded bins in an instant of time. Notice that, after a move, j^* may need to be updated to point to another bin.

We define three *regions* a, b, c, where a is the region bounded by load greater than $C - s_{k-i}$ ("above" L_i), b is the region bounded by load greater than $C - s_{k-i+1}$ and at most $C - s_{k-i}$ (same as L_i), and c is the region bounded by load between 0 and $C - s_{k-i+1}$ ("below" L_i). Let $\overline{W} = \sum_{j:j \text{ is underloaded}} \ell_j$, note that $0 \leq \overline{W} \leq nw_{\max}$. We will prove the following: $\Delta = \overline{W} - \ell_{j^*}$ decreases after a $b \to b$ move (i.e., $L_i \to L_i$), increases at most $nw_{\max}$ times due to $a \to a$ moves, and cannot increase by the other moves ($a \to b, a \to c, b \to a, b \to c, c \to a, c \to b, c \to c$). Consequently, after $O(nw_{\max})$ moves $L_i \to L_i$, Δ reaches its minimum and therefore no more moves $L_i \to L_i$ can occurs.

Fact 3.6 *Items with size greater than s_{k-i} do not move in moves of type $L_i \to L_i$. Therefore, due to the best-response, a move of type $L_i \to L_i$ always assign an item to one of the heaviest underloaded bins.*

In a $L_i \to L_i$ move, j^* remains underloaded or becomes an overloaded bin. If it remains underloaded, then, due to Fact 3.6, ℓ_{j^*} increases by at least one, and $\overline{W}$ do not change, thus, Δ decreases by at least one. Otherwise, if j^* becomes overloaded, then $\overline{W}$ decreases by at least $C - s_{k-i}$ and j^* will point to another underloaded bin with load at least 1, thus ℓ_{j^*} decreases by at most $C - s_{k-i} - 1$, therefore, Δ decreases by at least one.

For $a \to a$ moves, suppose that item i goes from bin j^- to bin j^+. If $\ell_{j^-} - w_i \geq C - s_{k-i}$ then Δ do not increases. Otherwise, Δ may increase as now $\overline{W}$ considers bin j^- in the sum. In the other hand, W_i increases and, from Lemma 3.1, W_i increases at most $nw_{\max}$. Therefore Δ also increases at most $nw_{\max}$ times.

After a move of type $a \to b$ or $a \to c$, if the load of ℓ_{j^*} decreases, then it decreases by at most the value decreased in $\overline{W}$, and therefore Δ do not increase. In moves of type $b \to a$ and $c \to a$, $\overline{W}$ decreases by the size of the moved item, and after the pointer j^* updates, we see that ℓ_{j^*} decreases by at most the size of the item moved. Therefore Δ do not increase. Moves of type $b \to c$, $c \to b$, and $c \to c$, can be analysed in a similar way as the previous cases. $\square$

Theorem 3.7 *In the bin packing game using the best-response strategy, the*

Nash equilibrium is reached in $O(mw_{\max}^2 + nw_{\max})$ *steps.*

Proof. The result follows from Lemmas 3.2, 3.3, 3.4 and 3.5 (noting that moves of type (1) are $L_0 \to L_0$). $\square$

Theorem 3.8 *In the bin packing game using the best-response strategy, the Nash equilibrium is reached in* $O(nkw_{\max})$ *steps.*

Proof. Notice that the moves of type $H_{i-1} \to H_{i-1}$ are one of these types: $L_i \to L_i$, $L_i \to H_i$, $H_i \to L_i$ and $H_i \to H_i$. Let $|L_i \to L_i|$ be the number of $L_i \to L_i$ moves; $|L_i \to H_i|$, $|H_i \to L_i|$, $|H_i \to H_i|$ are defined similarly. Thus, we can write the recurrence

$$\begin{aligned} |H_{i-1} \to H_{i-1}| &= |L_i \to L_i| + |L_i \to H_i| + |H_i \to L_i| + |H_i \to H_i| \\ |H_{k-1} \to H_{k-1}| &\leq mw_{\min} \end{aligned}$$

where $|H_{k-1} \to H_{k-1}| \leq mw_{\min}$ because each move increases the potencial by at least $w_{\min}$, and the maximum potencial above the "imaginary line" $C - w_{\min}$ is bounded by $mw_{\min}^2$ (see [1,3]). The total number of moves is bounded by $|H_{-1} \to H_{-1}|$. Solving the recurrence, we have $|H_{-1} \to H_{-1}| = \sum_{i=0}^{k-1} |L_i \to L_i| + \sum_{i=0}^{k-1} |L_i \to H_i| + \sum_{i=0}^{k-1} |H_i \to L_i| + mw_{\min}$. In what follows, we bound each of these sums. Similar to proof of Lemma 3.1, W_i never decreases and $W_i \leq ms_{k-i}$. In each move of type $L_i \to H_i$, W_i increases and therefore we have at most ms_{k-i} moves of this type (see proof of Lemma 3.2). Thus, $\sum_{i=0}^{k-1} |L_i \to H_i| \leq mS$, where $S = \sum_{i=1}^{k} s_i$. In each move of type $H_i \to L_i$, W_i increases and therefore we have at most ms_{k-i} moves of this type (see proof of Lemma 3.3). Thus, $\sum_{i=0}^{k-1} |H_i \to L_i| \leq mS$. Using Lemma 3.5, $\sum_{i=0}^{k-1} |L_i \to L_i| \leq knw_{\max}$. Adding all the terms, the result follows. $\square$

References

[1] Bilò, V., *On the packing of selfish items*, 20th IEEE International Parallel and Distributed Processing Symposium, IPDPS (2006), 9–18.

[2] Epstein, L. and E. Kleiman, *Selfish bin packing*, 16th European Symposium on Algorithms, ESA (2008), 368–380.

[3] Even-Dar, E. and A. Kesselman and Y. Mansour, *Convergence time to Nash equilibrium in load balancing*, ACM Trans. on Algorithms. **3** 3 (2007), 32-es.

[4] Miyazawa, F. K. and A. L. Vignatti, *Distributed Selfish Bin Packing*, 24th IEEE International Parallel and Distributed Processing Symposium, IPDPS (2009), 1–8.

[5] Yu, G. and G. Zhang, *Bin Packing of Selfish Items*, 4th International Workshop on Internet and Network Economics, WINE (2008), 446–453.